Algebraically closed real geodesics on n–dimensional ellipsoids are dense in the parameter space and related to hyperelliptic tangential coverings

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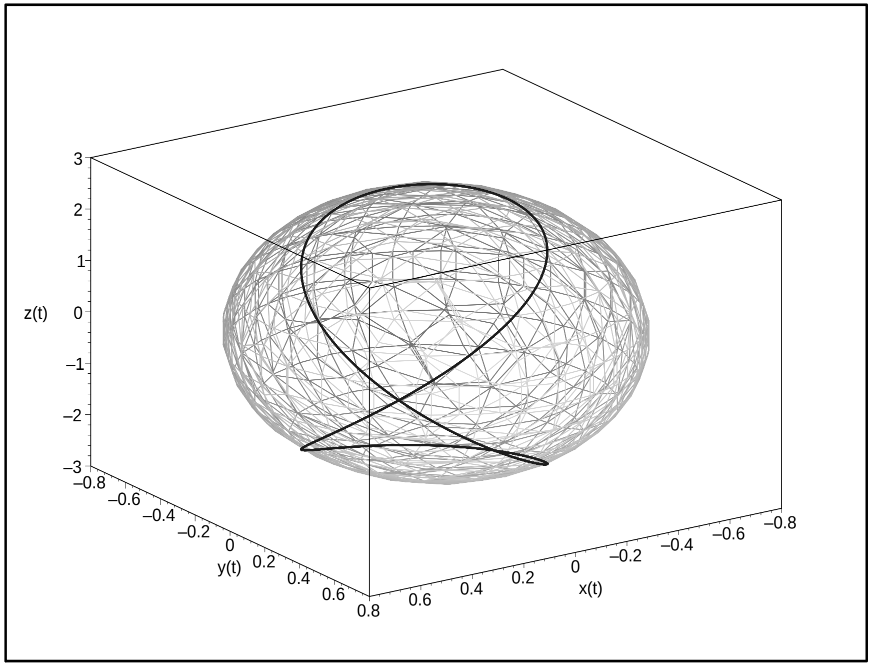
Nonlinear Physics Theory and Experiment V

Aim:



To characterize algebraically a dense set of real closed geodesics on n dimensional ellipsoids

Introduction:

• The Jacobi problem of the geodesic motion on an n-dimensional ellipsoid

$$\mathcal{Q} \; : \; \left\{ \frac{\mathbf{x_1^2}}{\mathbf{a_1}} + \cdots + \frac{\mathbf{x_{n+1}^2}}{\mathbf{a_{n+1}}} = \mathbf{1} \right\}$$

is well known to be integrable and to be linearized on a covering of the Jacobian of a genus $\mathbf{n}$ hyperelliptic curve $\boldsymbol{\Gamma}$.

• Let $\lambda_{\mathbf{1}}, \ldots, \lambda_{\mathbf{n}}$ be the ellipsoidal coordinates on $\mathcal{Q}$ defined by the formulas

$$\mathbf{x_i} = \sqrt{\frac{(\mathbf{a_i} - \lambda_{\mathbf{1}}) \cdots (\mathbf{a_i} - \lambda_{\mathbf{n}})}{\prod_{\mathbf{j} \neq \mathbf{i}} (\mathbf{a_i} - \mathbf{a_j})}}, \qquad \mathbf{i} = \mathbf{1}, \ldots, \mathbf{n} + \mathbf{1}.$$

• Fix the constants of motion $\mathbf{H} = \mathbf{1}, \mathbf{c_1}, \ldots, \mathbf{c_{n-1}}$

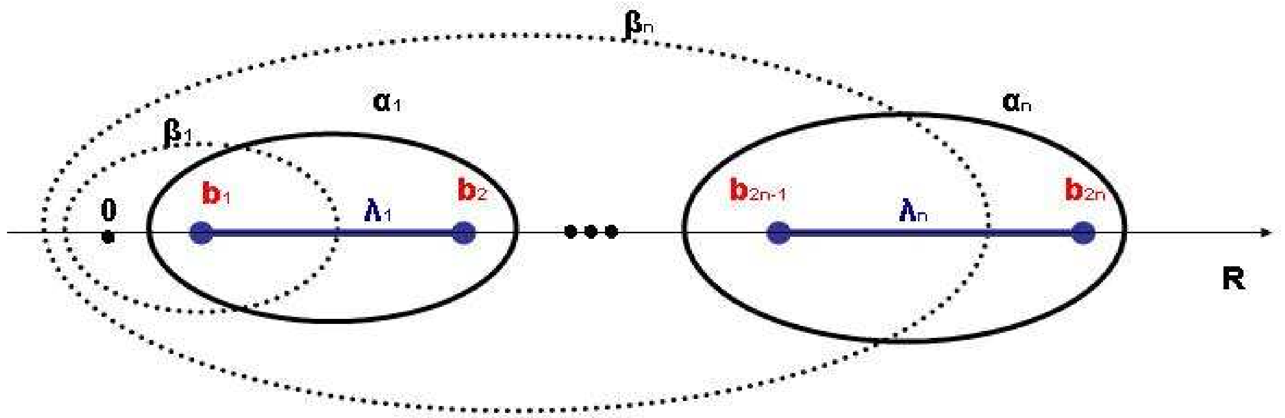
• In the new length parameter s:

$$\mathbf{dl} = \lambda_{\mathbf{1}} \cdots \lambda_{\mathbf{n}} \, \mathbf{ds},$$

the evolution of the ellipsoidal coordinates $\lambda_{\mathbf{k}}$ is described by quadratures (Abel map)

$$\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} \int_{\mathbf{P_0}}^{\mathbf{P_k}} \begin{pmatrix} \omega_{\mathbf{1}} \\ \vdots \\ \omega_{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} \\ \vdots \\ \mathbf{u_n} \end{pmatrix} = \begin{pmatrix} \mathbf{t} + \mathbf{cost} \\ \vdots \\ \mathbf{const.} \end{pmatrix}$$

which involve n independent holomorphic differentials $\omega_{\mathbf{j}} = \dfrac{\lambda^{\mathbf{k-1}} \mathbf{d}\lambda}{\mathbf{w}}, \quad \mathbf{j} = \mathbf{1}, \ldots, \mathbf{n}$, on the genus $\mathbf{n}$ hyperelliptic curve $\boldsymbol{\Gamma}$



$$\Gamma := \{ w^2 = -\lambda \prod_{j=1}^{n-1} (\lambda - c_j) \prod_{i=1}^{n+1} (\lambda - a_i) = - \prod_{j=0}^{2n} (\lambda - b_j) \}$$

$$\{ 0, a_1 < \cdots < a_{n+1}, c_1 < \cdots < c_{n-1} \} = \{ b_0 = 0 < b_1 < \cdots < b_{2n} \}.$$

Reality conditions:

- $\boxed{c_i = b_{2i}}$ o $\boxed{c_i = b_{2i+1}}$, $i = 1, \ldots, n-1$ ($\mathcal{Q}$ ellipsoid)
- $\boxed{b_{2i-1} < \lambda_i < b_{2i}}$ $i = 1, \ldots, n$ (real geodesic on $\mathcal{Q}$)

Real closed geodesics

A real geodesic on $\mathcal{Q}$ is closed if and only if there exist a non trivial cycle $\alpha = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} \mathbf{m_i}\alpha_{\mathbf{i}}$ and $\mathbf{T} > \mathbf{0}$ such that

$$\oint_\alpha \omega_{\mathbf{1}} = \mathbf{T}, \qquad \oint_\alpha \omega_{\mathbf{j}} = \mathbf{0}, \quad \mathbf{j} = \mathbf{2}, \ldots, \mathbf{n}, \quad (1)$$

- Closedness condition (1) is transcendental in the parameters $\mathbf{a_1}, \ldots, \mathbf{a_{n+1}}$, $\mathbf{c_1}, \ldots, \mathbf{c_{n-1}}$.

- If a geodesic on $\mathcal{Q}$ is closed, then all the geodesics on $\mathcal{Q}$ sharing the same constants of motion $\mathbf{c_1}, \ldots, \mathbf{c_{n-1}}$ are closed.

- (density) For all $\mathbf{a_1} < \cdots < \mathbf{a_{n+1}}$ and for any $\mathbf{n}$-tuple $\zeta_{\mathbf{i}}$, $\mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}$ such that $\mathbf{a_1} \le \zeta_{\mathbf{1}} < \mathbf{a_2} < \cdots < \mathbf{a_n} \le \zeta_{\mathbf{n}} < \mathbf{a_{n+1}}$, there is a dense set $\mathbf{I} \subset [\zeta_{\mathbf{1}}, \zeta_{\mathbf{2}}] \times \cdots \times [\zeta_{\mathbf{n-1}}, \zeta_{\mathbf{n}}]$ (in the natural topology of $\mathbb{R}^{\mathbf{n-1}}$), such that $\forall(\mathbf{c_1}, \ldots, \mathbf{c_{n-1}}) \in \mathbf{I}$, there exist nontrivial integers $\mathbf{m_i}, \mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}$ and a real $\mathbf{T} > 0$ such that (1) holds.

Under which conditions, (1) is algebraic in the parameters $\mathbf{a_1, \ldots, a_{n+1}}$, $\mathbf{c_1, \ldots, c_{n-1}}$?

The real geodesics on the ellipsoid $\mathcal{Q} = \left\{ \frac{\mathbf{x_1^2}}{\mathbf{a_1}} + \cdots + \frac{\mathbf{x_{n+1}^2}}{\mathbf{a_{n+1}}} = \mathbf{1} \right\}$ with constants of motion $\mathbf{c_1, \ldots, c_{n-1}}$ are closed and the closedness condition

$$\oint_\alpha \omega_\mathbf{1} = \mathbf{T}, \qquad \oint_\alpha \omega_\mathbf{j} = \mathbf{0}, \quad \mathbf{j} = \mathbf{2}, \ldots, \mathbf{n}, \qquad (1)$$

is algebraic in the parameters $\mathbf{a_1, \ldots, a_{n+1}}$, $\mathbf{c_1, \ldots, c_{n-1}}$ if and only if

the associated hyperelliptic curve $\mathbf{\Gamma}$ with marked point $\mathbf{P_0} = (\mathbf{0}, \mathbf{0})$ is a **real hyperelliptic tangential cover**.

- Sufficiency: [Fedorov, 2005] and [A-Fedorov 2006]
- Necessity: [A-2007]

Algebraically closed real geodesics [A-2007]

The real geodesics on $\mathcal{Q}$ are algebraically closed if and only if **both the real and the imaginary geodesics on $\mathcal{Q}$ are closed**, i.e. there exist two cycles $\alpha = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{m_i} \alpha_\mathbf{i}$ (real) and $\beta = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{m'_i} \beta_\mathbf{i}$ (imaginary), and $\mathbf{T}, \mathbf{T}' > \mathbf{0}$ such that

$$\begin{aligned} &\oint_\alpha \omega_\mathbf{1} = \mathbf{T}, \qquad &&\oint_\beta \omega_\mathbf{1} = \sqrt{-\mathbf{1}}\mathbf{T}', \\ &\oint_\alpha \omega_\mathbf{j} = \mathbf{0}, \qquad &&\oint_\beta \omega_\mathbf{j} = \mathbf{0}, \quad \mathbf{j} = \mathbf{2}, \ldots, \mathbf{n}. \end{aligned} \tag{2}$$

Since $\omega_\mathbf{2}, \ldots, \omega_\mathbf{n}$ are form a maximal system of holomorphic differentials vanishing at the Weierstrass point $\mathbf{P_0} = (\mathbf{0}, \mathbf{0})$ and since (2) implies that $\mathrm{Jac}(\mathbf{\Gamma})$ is isogenous to $\mathcal{E} \times \mathcal{A}_\mathbf{n-1}$, (2) is equivalent to require the existence of a morphism $\pi : (\mathbf{\Gamma}, \mathbf{P_0}) \mapsto (\mathcal{E}, \mathcal{Q}_\mathbf{0})$ (hyperelliptic tangential cover in the sense of Treibich-Verdier). The covering is **real** in the sense that $\mathcal{E}$ is a real elliptic curve (all real branch points).

Closed geodesics and x–periodic solutions to KdV

Thanks to the Moser–Trubowitz isomorphism, there exists a natural link between geodesics on quadrics and stationary solutions to the Korteweg-de Vries equation

$$u_t = 6uu_x - u_{xxx}$$

Algebraic-geometric solutions to KdV corrispond to geometric data ($\mathbf{\Gamma}$ hyperelliptic curve, ...).

A KdV solution $u(x,t)$ is an elliptic soliton if

$$u(x,t) = 2\sum_{j=1}^{N} \mathcal{P}(x - q_j(t)) + const.,$$

with $\mathcal{P}(x)$ the Weierstrass P-function, where $q_j(t)$ satisfy the H. Airault,J. Moser, H.P. McKean (1977) condition

$$\sum_{j=1, j\neq k}^{N} \mathcal{P}'(q_j(t) - q_k(t)) = 0, \quad k = 1, \ldots, N. \quad \text{(KdV locus)}$$

$u(x,t)$ is a KdV elliptic soliton if and only if $\mathbf{\Gamma}, \mathbf{P_0}$ is a hyperelliptic tangential cover (A. Treibich- J.L. Verdier).

A partial topological classification of such coverings and the corresponding classification of KdV (and KP) elliptic potentials has been successfully accomplished by A. Treibich e J.L. Verdier in the complex moduli space of hyperelliptic (algebraic) curves.

E. Colombo, G.P. Pirola e E. Previato (1994) proved that the hyperelliptic tangential coverings of genus $\mathbf{n}$ are dense in the complex moduli space of the hyperelliptic curves of genus $\mathbf{n}$.

Does there exist a dense subset in the real parameter space $\{\mathbf{a_1}, \ldots, \mathbf{a_{n+1}}, \mathbf{c_1}, \ldots, \mathbf{c_{n-1}}\}$ corresponding to algebraically closed real geodesics on n-dimensional ellipsoids?

Density of the real algebraic closed geodesics [A-07]

For the elliptic KdV solutions, using Riemann bilinear relations, H.P. McKean and P. van Moerbeke (1980) construct a locally invertible real analytic map from $(\mathbf{b_1}, \ldots \mathbf{b_{2n}}) \in \mathbb{R}^{2n}$ (space of the real ramification points of $\mathbf{\Gamma}$ = space of the parameters for the geodesic problem) to $(\mathbf{X}, \mathbf{Y}) \in \mathbb{R}^{\mathbf{2n}}$ such that

$$\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} \mathbf{X_i} \oint_{\alpha_{\mathbf{i}}} \omega_{\mathbf{k}} = \begin{cases} \mathbf{1}, & \mathbf{k} = \mathbf{1}, \\ \mathbf{0}, & \mathbf{k} = \mathbf{2}, \ldots, \mathbf{n}, \end{cases}$$

$$\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} \mathbf{Y_i} \oint_{\beta_{\mathbf{i}}} \omega_{\mathbf{k}} = \begin{cases} \sqrt{-\mathbf{1}}, & \mathbf{k} = \mathbf{1}, \\ \mathbf{0}, & \mathbf{k} = \mathbf{2}, \ldots, \mathbf{n}. \end{cases}$$

Comparing the above relation with the condition of (algebraically) closed real geodesics, it is possible to verify the following density theorem:

Theorem [A07] *Let* $\mathbf{a_1 < ... < a_{n+1}, c_1 < ... < c_{n-1}}$, *such that the real geodesics on the ellipsoid* $\mathcal{Q} = \{\mathbf{x_1^2/a_1 + \cdots + x_{n+1}^2/a_{n+1} = 1}\}$ *with constants of motion* $\mathbf{c_j}$, *are closed. Then,* $\forall \epsilon > \mathbf{0}$ *there exist* $\mathbf{a_1^\epsilon < \cdots < a_{n+1}^\epsilon, c_1^\epsilon < \cdots < c_{n-1}^\epsilon}$ *such that*

$$\sum_{\mathbf{j=1}}^{\mathbf{n-1}} (\mathbf{c_j} - \mathbf{c_j^\epsilon})^{\mathbf{2}} + \sum_{\mathbf{i=1}}^{\mathbf{n+1}} (\mathbf{a_i} - \mathbf{a_i^\epsilon})^{\mathbf{2}} < \epsilon$$

and the real geodesics on $\mathcal{Q}^\epsilon = \{\mathbf{x_1^2/a_1^\epsilon + \cdots + x_{n+1}^2/a_{n+1}^\epsilon = 1}\}$ *with constants of motion* $\mathbf{c_j^\epsilon}$, *are algebraically closed (*$\mathbf{\Gamma}^\epsilon$ *is a real hyperelliptic tangential covering).*

Optimal density characterization: how many parameters may be kept fixed if one wants to preserve the character of the initial geodesics (value of the period map and/or length of the real closed geodesic)?

The period mapping

For simplicity, let us restrict ourselves to the case of closed geodesics on triaxial ellipsoids ($\mathbf{n = 2}$).

The period mapping

$$\mathbf{c} \mapsto \varphi(\mathbf{c}) = \begin{cases} \mathbf{2} \oint_{\alpha_\mathbf{2}} \omega_\mathbf{2} : \mathbf{2} \oint_{\alpha_\mathbf{1}} \omega_\mathbf{2}, & \mathbf{a_1} < \mathbf{c} < \mathbf{a_2} < \mathbf{a_3}, \\ \\ \mathbf{2} \oint_{\alpha_\mathbf{1}} \omega_\mathbf{2} : \mathbf{2} \oint_{\alpha_\mathbf{2}} \omega_\mathbf{2}, & \mathbf{a_1} < \mathbf{a_2} < \mathbf{c} < \mathbf{a_3}, \end{cases}$$

measures the ratio between oscillations and windings of a geodesic with constant of motion $\mathbf{c}$.

A real geodesic on $\mathcal{Q}$ with parameter $\mathbf{c}$ is closed if and only if $\varphi(\mathbf{c}) \in \mathbb{Q}$.

Optimal characterization of the density property $(n \geq 2)$ [A-2007]

Theorem *Let* $\mathbf{\Gamma} = \{\mu^{\mathbf{2}} = -\lambda(\lambda - \mathbf{c}) \prod_{\mathbf{i=1}}^{\mathbf{3}} (\lambda - \mathbf{a_i})\}$*, such that the real geodesics on the ellipsoid* $\mathcal{Q} = \{\mathbf{x_1^2}/\mathbf{a_1} + \mathbf{x_2^2}/\mathbf{a_2} + \mathbf{x_3^2}/\mathbf{a_3} = \mathbf{1}\}$ *with constant of motion* $\mathbf{c}$ *are closed, with period mapping value* φ *and length* $\mathbf{T}$.

Then, there exists a sequence $\{\mathbf{a_1^{(k)}}, \mathbf{a_2^{(k)}}, \mathbf{c^{(k)}}\} \in \mathbb{R}^{\mathbf{3}}$ *such that* $\lim_{\mathbf{k}\to+\infty} \mathbf{c^{(k)}} = \mathbf{c}$, $\lim_{\mathbf{k}\to+\infty} \mathbf{a_i^{(k)}} = \mathbf{a_i}$, $\mathbf{i} = \mathbf{1}, \mathbf{2}$ *and such that the real geodesics on* $\mathcal{Q}^{(\mathbf{k})} = \{\mathbf{x_1^2}/\mathbf{a_1^{(k)}} + \mathbf{x_2^2}/\mathbf{a_2^{(k)}} + \mathbf{x_3^2}/\mathbf{a_3} = \mathbf{1}\}$ *with parameter* $\mathbf{c^{(k)}}$ *are algebraically closed, with the same length* $\mathbf{T^{(k)}} = \mathbf{T}$ *and the same value of the period mapping* $\varphi^{(\mathbf{k})} = \varphi$.

Remark: if we require to preserve only the period mapping, then it is possible to keep fixed to parameters (e.g. $\mathbf{a_1}, \mathbf{a_3}$).

The period mapping and the second covering [A-07]

The case of the triaxial ellipsoid is peculiar since the hyperelliptic curve has genus 2.

Theorem (E. Colombo *et al.* 1994) *Let* $\mathbf{\Gamma}$ *a genus 2 curve which covers an elliptic curve* $\pi_{\mathbf{1}} : \mathbf{\Gamma} \mapsto \mathcal{E}_{\mathbf{1}}$ *and let* $\pi_{\mathbf{2}} : \mathbf{\Gamma} \mapsto \mathcal{E}_{\mathbf{2}}$ *be another covering such that* $\mathrm{Jac}(\mathbf{\Gamma})$ *is isogenous to* $\mathcal{E}_{\mathbf{1}} \times \mathcal{E}_{\mathbf{2}}$*. Then* $\pi_{\mathbf{1}}$ *is tangential exactly at the points where* $\pi_{\mathbf{2}}$ *is ramified.*

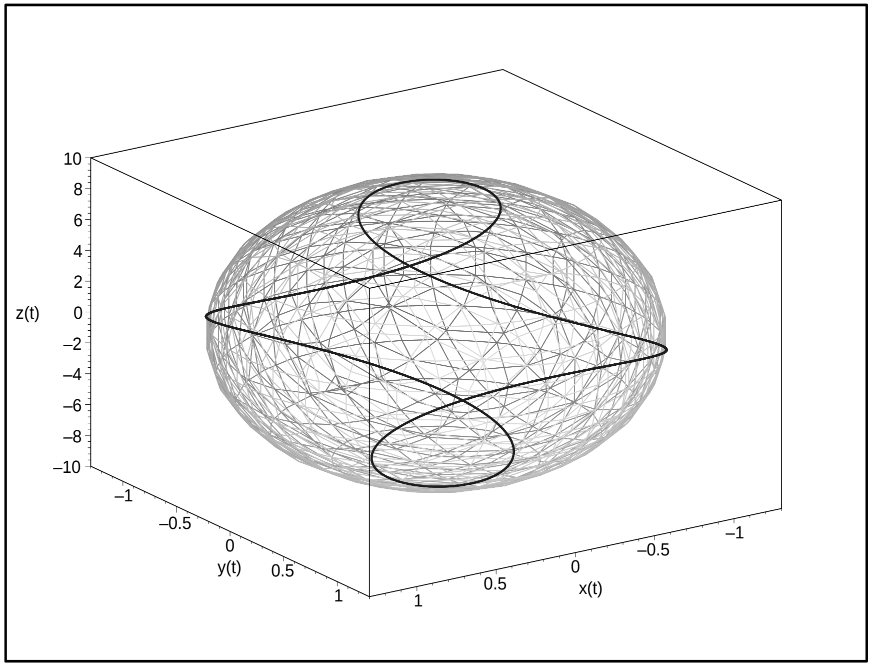
For the case of algebraically closed real geodesics on the triaxial ellipsoid $\mathcal{Q}$, $\pi_{\mathbf{2}} : \mathbf{\Gamma} \mapsto \mathcal{E}_{\mathbf{2}}$ is ramified (of order 3) at $\mathbf{P_0} = (0, 0)$ and the pull-back of the holomorphic differential $\mathbf{\Omega}$ di $\mathcal{E}_{\mathbf{2}}$ is $\pi_{\mathbf{2}}^* \mathbf{\Omega} = \omega_{\mathbf{2}}$.

The topological character of the second covering allows to reduce the computation of the period mapping of an algebraically closed geodesic to the solution of algebraic equations associated to the second covering (A07).

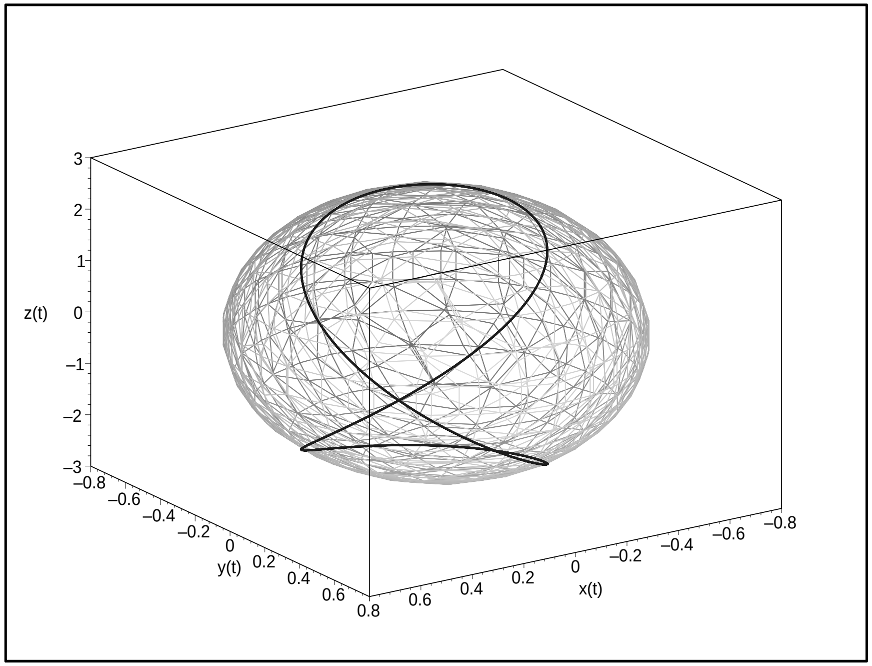
The topological character of a covering is a sequence of four integer numbers $(\nu_{\mathbf{0}}, \nu_{\mathbf{1}}, \nu_{\mathbf{2}}, \nu_{\mathbf{3}})$ which counts the Weierstrass points of $\mathbf{\Gamma}$ in the preimage of the four ramification points of $\mathcal{E}_{\mathbf{2}}$, except for $\mathbf{P_0} = (\mathbf{0}, \mathbf{0}) \in \mathbf{\Gamma}$ - the Weierstrass point in which the second covering is ramified.

For KdV elliptic solitons, the topological character of the second covering reduces the problem of describing the pole dynamics of the genus 2 KdV elliptic solitons to the search of solutions of certain algebraic equations related to the covering and to the inversion of elliptic integrals (Enolskii and Belokolos 1989, etc.)

Examples of algebraically closed geodesics



For geodesics on triaxial ellipsoids $\mathcal{Q}$ (the hyperelliptic $\mathbf{\Gamma}$ has genus 2), we have worked out the reality conditions for real algebraically closed geodesics when $\mathbf{d} = \mathbf{3}, \mathbf{4}, \mathbf{5}$ in the form of algebraic equations in the parameters $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ and $\mathbf{c}$ [AF06,A07].



• $\mathbf{d = 3}$ (Lamé potential, Hermite covering) The possible values of the period mapping are $\mathbf{2:1}$ or $\mathbf{1:2}$ [Fed05,A-07].

$\mathbf{0 < a_1 < a_2 < a_3}$ and $\mathbf{c} \in]\mathbf{a_1, a_3}[\backslash\{\mathbf{a_2}\}$ be given. Then the geodesics on the ellipsoid $\mathcal{Q} = \{\mathbf{x_1^2/a_1 + x_2^2/a_2 + x_3^2/a_3 = 1}$ with constant of the motion c are algebraically closed if and only if

$$\mathbf{\frac{1}{c^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} - 2\left(\frac{1}{ca_2} + \frac{1}{ca_3} + \frac{1}{a_2 a_3}\right) = 0,} \qquad (3)$$

$$\mathbf{a_1 = \frac{3ca_2a_3}{2(a_2a_3 + c(a_2 + a_3))}.} \qquad (4)$$

(3) and (4) may be inverted and we get a_1, a_3 parametrically in function of a_2, c or a_2, c in function of a_1, a_3.

- Let $a_2, c > 0$ be given and $a_2 \neq c$. Then (3)-(4) is equivalent to

$$a_3 = \left(\frac{1}{\sqrt{a_2}} - \frac{1}{\sqrt{c}}\right)^{-2}, \qquad a_1 = \frac{3a_2c}{4(a_2 + c - \sqrt{ca_2})}.$$

- Let $0 < a_1 < a_3$ be given. Then (3)-(4) is equivalent to

$$\frac{1}{a_2}, \quad \frac{1}{c} = \pm\frac{1}{2\sqrt{a_3}} + \sqrt{\frac{4}{3a_1} - \frac{3}{4a_3}}.$$