Integrability in $\mathcal{N} = 4$ SYM

a tool for QCD ?

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QCD vs $\mathcal{N}$-SYM

- In QCD, the gauge group is SU(3), fermions $\psi \sim \mathbf{3}$.

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\,\text{Tr}F^2 + i\,\overline{\psi}\gamma_\mu D^\mu\,\psi$$

- In $\mathcal{N}$ SYM, the gauge group is SU(N_c), **all** fields $X = (A_\mu, \lambda^A, \varphi^{AB}) \sim \mathbf{adj}$.
- internal symmetry SU($\mathcal{N}$), gauginos in the fundamental, scalars $\varphi^{AB} = -\varphi^{BA}$

$$\begin{aligned}\mathcal{L}_{\text{SYM}}^{\mathcal{N}=1,2,4} \quad &= \quad \text{Tr}\Big\{ -\frac{1}{2}F^2 + 2\,i\,\overline{\lambda}_A\,\sigma\cdot D\,\lambda^A + \frac{1}{2}D_\mu\varphi^{AB}\,D^\mu\overline{\varphi}_{AB} \\ &\quad -\boxed{h_Y}\,(\lambda^A\,[\overline{\varphi}_{AB},\lambda^B] + \text{c.c.}) - \boxed{h_4}\,[\varphi^{CD},\overline{\varphi}_{AB}][\varphi^{AB},\overline{\varphi}_{CD}]\Big\}.\end{aligned}$$

$AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ SYM

- **AdS**: il limite $\lambda \to \infty$ e' quello di stringa **classica**.

- **CFT**: teoria delle perturbazioni intorno a $\lambda = 0$.
- **overlap** nel limite BMN [Berenstein, Maldacena, Nastase, 02]

$$\lambda' = \frac{\lambda}{J^2} = \text{fissato}, \qquad J \to \infty$$

- Integrabilita' classica su $AdS_5 \times S^5$ [Bena, Polchinski, Roiban, 03]

$$\Downarrow$$

Integrabilita' a livello di Δ_{CFT}

Integrabilita' e limite planare

- Operatori **quasipartonici** sul cono-luce [Korchemsky at al., 04]

$$\mathbb{O}_{\text{adj}}(z_1,\dots,z_L) = \boxed{\textbf{Tr}}\{X(z_1\, n)\cdots X(z_L\, n)\},$$
$$\mathbb{O}_{\text{fun}}(z_1,\dots,z_{N_c}) = \varepsilon_{i_1,\dots,i_{N_c}} X^{i_1}(z_1\, n)\cdots X^{i_{N_c}}(z_{N_c}\, n).$$

- Dall'analisi di operatori a 3 gaugini (max. hel.) a due loops in $\mathcal{N}=1,2,4$ [Belitski at al., 05]
  - 1 loop $\mathfrak{D}$ **universale** $\longrightarrow$ catena integrabile XXX_{-s}
  - 2 loops, QCD, integrabilita' rotta dai termini non planari
  - 2 loops, SYM, integrabilita' **non-rotta** da $\beta \neq 0$

    $\mathfrak{D}$ dipende semplicemente da $\mathcal{N}$

Conclusione: integrabilita' legata al limite planare.

- Il settore di gauge e' **universale**

Il contenuto di *materia* puo' essere descritto **uniformemente**

- $\boxed{\mathcal{N}=1}$ 1 gaugino, nessuno scalare

$$h_Y = 0, \qquad h_4 = 0.$$

- $\boxed{\mathcal{N}=2}$ 2 gaugini, 1 scalare complesso

$$h_Y = g, \qquad h_4 = \frac{g^2}{16}, \qquad \varphi^{AB} = \sqrt{2}\,\varepsilon^{AB}\,\varphi.$$

- $\boxed{\mathcal{N}=4}$ 4 gaugini, 3 scalari complessi

$$h_Y = \sqrt{2}\,g, \qquad h_4 = \frac{g^2}{8}, \qquad \varphi^{AB} = \frac{1}{2}\,\varepsilon^{ABCD}\,\overline{\varphi}_{CD}.$$

- L'invarianza conforme e' regolata dalla β function

$$\beta(g^2) = -\beta_0 \frac{g^2}{8\pi^2} - \beta_1 \left(\frac{g^2}{8\pi^2}\right)^2 + \cdots ,$$

- QCD, $\beta \neq 0$, invarianza conforme solo ad 1 loop
- in SYM,

$$\beta_{\text{SYM}}(g^2) = -(4-\mathcal{N}) \frac{g^2 N_c}{8\pi^2} - (4-\mathcal{N})(2-\mathcal{N}) \left(\frac{g^2 N_c}{8\pi^2}\right)^2 + \cdots .$$

inoltre

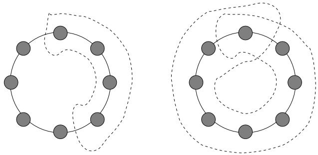
$$\beta_{\text{SYM}}^{\mathcal{N}=2} = \beta^{1\,\text{loop}}, \qquad \beta_{\text{SYM}}^{\mathcal{N}=4} = 0$$

$\left\{ \begin{array}{l} \text{invarianza conforme} \\ \text{SUSY} \\ \text{limite planare} \end{array} \right.$ chi e' **responsabile** dell'integrabilita' ?

The wrapping problem

Higher order Bethe Ansatz is asymptotic !

- BA equations are asymptotic: correct up to $\mathcal{O}(g^{2L})$ for length-L operators
- wrapping: interactions with growing range !



- Sector dependent details

Example: Konishi operator $\mathcal{K} = \mathrm{Tr}\bar{\Phi}\Phi \notin \mathfrak{su}(2)$

Descendant in $\mathfrak{su}(2)$: $\mathcal{K}' = \mathrm{Tr}[Z, X][Z, X]$

Wrapping starts at 4 loops !

- The 4 loops anomalous dimension is not known !

$$\gamma = 4 + 12\,g^2 - 48\,g^4 + 336\,g^6 + \gamma_4\,g^8 + \cdots$$

$$\gamma_4 = -16 \times 318$$ [Rej, Serban, Staudacher, 05]

$$\gamma_4 = -16\,\left(\frac{705}{4} + 18\,\zeta_3\right)$$ [Beisert, Eden, Staudacher, 07]

$$\gamma_4 = -\left(\frac{5307}{2} + 564\,\zeta_3\right)$$ [Staudacher, Lipatov et al., 07]

$$\gamma_4 = -2584 + 384\,\zeta_3 - 1440\,\zeta_5$$ [Zanon et al., 07]

$$\gamma_4 = -(2607 + 28\,\zeta_3 + 140\,\zeta_5)$$ [Mann, Keeler, 08]

Bethe Ansatz in SYM, cronologia

- 1-loop, $\mathfrak{su}(2) \oplus \mathfrak{so}(6)$ [Minahan, Zarembo, 02]
- 1-loop, $\mathfrak{psu}(2,2|4)$ [Beisert, Staudacher, 03]
- 3-loop, $\mathfrak{su}(2)$ [Serban, Staudacher, 04]
- ∞-loop, asintotico, $\mathfrak{su}(2)$ [Beisert, Dippel, Staudacher, 04]
- ∞-loop, asintotico, $\mathfrak{psu}(2,2|4)$ [Beisert, Staudacher, 05]
- ∞-loop, **non asintotico ?**, $\mathfrak{su}(2)$ [Rej, Serban, Staudacher, 05]
  (incompleto ma intrigante)
- Dressing, $\mathfrak{psu}(2,2|4)$ [Beisert, Eden, Staudacher, 07]

- Variabili spettrali

$$x(u) = \frac{u}{2}\left(1+\sqrt{1-\frac{2\,g^2}{u^2}}\right) \qquad \leftrightarrow \qquad u = x + \frac{g^2}{2\,x},$$

$$x^{\pm}(u) = x\left(u \pm \frac{i}{2}\right) = u \pm \frac{i}{2} + \mathcal{O}(g^2)$$

- cariche conservate

$$Q_r = \frac{1}{r-1}\sum_{j=1}^{K_4}\left(\frac{i}{(x_{4,j}^+)^{r-1}} - \frac{i}{(x_{4,j}^-)^{r-1}}\right),$$

$$\gamma = \delta D = g^2\,Q_2 = g^2\sum_{j=1}^{K_4}\left(\frac{i}{x_{4,j}^+} - \frac{i}{x_{4,j}^-}\right).$$

- Long range Bethe equations

$$
\begin{aligned}
1 &= \prod_{j=1}^{K_4} \frac{x^+_{4,j}}{x^-_{4,j}}, \\
1 &= \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}\,\eta_1}{u_{1,k} - u_{2,j} - \frac{i}{2}\,\eta_1} \prod_{j=1}^{K_4} \frac{1 - g^2/2\, x_{1,k}\, x^{+\eta_1}_{4,j}}{1 - g^2/2\, x_{1,k}\, x^{-\eta_1}_{4,j}}, \\
1 &= \prod_{\substack{j=1 \\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i\,\eta_1}{u_{2,k} - u_{2,j} + i\,\eta_1} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}\,\eta_1}{u_{2,k} - u_{3,j} - \frac{i}{2}\,\eta_1} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}\,\eta_1}{u_{2,k} - u_{1,j} - \frac{i}{2}\,\eta_1}, \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}\,\eta_1}{u_{3,k} - u_{2,j} - \frac{i}{2}\,\eta_1} \prod_{j=1}^{K_4} \frac{x_{3,k} - x^{+\eta_1}_{4,j}}{x_{3,k} - x^{-\eta_1}_{4,j}},
\end{aligned}
$$

contd.

$$1 = \left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{x_{4,k}^{+\eta_1} - x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2} - x_{4,j}^{+\eta_2}} \, \frac{1 - g^2/2\, x_{4,k}^+ \, x_{4,j}^-}{1 - g^2/2\, x_{4,k}^- \, x_{4,j}^+} \, \sigma^2(x_{4,k}, x_{4,j}) \right)$$

$$\times \prod_{j=1}^{K_4} \frac{1 - g^2/2\, x_{4,k}^{-\eta_1}\, x_{1,j}}{1 - g^2/2\, x_{4,k}^{+\eta_1}\, x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^{-\eta_1} - x_{3,j}}{x_{4,k}^{+\eta_1} - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^{-\eta_2} - x_{5,j}}{x_{4,k}^{+\eta_2} - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - g^2 \ldots}{1 - g^2 \ldots}$$

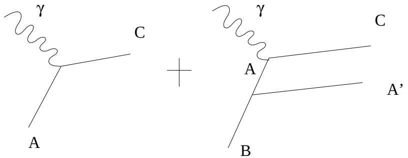
$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}\,\eta_2}{u_{5,k} - u_{6,j} - \frac{i}{2}\,\eta_2} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^{+\eta_2}}{x_{5,k} - x_{4,j}^{-\eta_2}},$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i\,\eta_2}{u_{6,k} - u_{6,j} + i\,\eta_2} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}\,\eta_2}{u_{6,k} - u_{5,j} - \frac{i}{2}\,\eta_2} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + \frac{i}{2}\,\eta_2}{u_{6,k} - u_{7,j} - \frac{i}{2}\,\eta_2},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}\,\eta_2}{u_{7,k} - u_{6,j} - \frac{i}{2}\,\eta_2} \prod_{j=1}^{K_4} \frac{1 - g^2/2\, x_{7,k}\, x_{4,j}^{+\eta_2}}{1 - g^2/2\, x_{7,k}\, x_{4,j}^{-\eta_2}}.$$

Proprieta' generali delle splitting functions ad 1 loop

- La parte dovuta all'emissione reale e' calcolabile dal processo elementare



- Senza fattori legati a $SU(N)$, $P_{ab} \sim V_{ab}$

$$\begin{aligned} \textbf{crossing} \quad & V_{BA}^{(C)}(x) = V_{CA}^{(B)}(1-x), \\ \textbf{Gribov-Lipatov} \quad & V_{AB}(x) = (-1)^{2s_A+2s_B-1}\, x\, V_{BA}(x^{-1}), \\ \textbf{SUSY} \quad & V_{qq} + V_{gq} = V_{qg} + V_{gg}. \end{aligned}$$

- **Tutte** le V_{ab} si ottengono, per $x < 1$ da V_{qq}

Twist-2: Ereditarieta' LBK a livello di $\mathcal{P}$

- Sviluppando $\mathcal{P}(N;\alpha)$ nel coupling fisico / cusp anomaly

$$\alpha_{\rm ph} = \alpha\,(1 - \frac{1}{2}\,\zeta_2\,\alpha + \frac{11}{20}\,\zeta_2^2\,\alpha^2 + \cdots)$$

- Tutte le correzioni logaritmiche sono **ereditate** da 1 loop

$$\begin{aligned}
\mathcal{P}_1 &= -S_1 \qquad \boxed{\sim \log N} \\
\mathcal{P}_2 &= \frac{1}{2}\,\widehat{S}_3 - \frac{1}{2}\widehat{Y}_{-3} + B_2, \\
\mathcal{P}_3 &= -\frac{1}{2}\,\widehat{S}_5 + \frac{3}{2}\widehat{Y}_{-5} + B_3 + \zeta.\,\frac{1}{2}\widehat{S}_3 \\
&\quad + \boxed{S_1} \cdot [\widehat{Y}_{-4} - \frac{1}{2}(\widehat{S}_{-4} + \widehat{S}^2_{-2} + \zeta_2 \cdot \frac{1}{2}\widehat{S}_{-2}] \quad \boxed{\sim \frac{\log N}{N^2}}
\end{aligned}$$

- Qualche struttura iterativa e' visibile

Twist-3: Ereditarieta' LBK a livello di $\mathcal{P}$

- Sviluppando $\mathcal{P}$ nel coupling fisico

$$g_{\rm ph}^2 = \frac{N_c\,\alpha_{\rm ph}}{2\,\pi} = g^2 - \zeta_2\,g^4 + \frac{11}{5}\zeta_2^2\,g^6 - \left(\frac{73}{10}\zeta_2^3 + \zeta_3^2\right)\,g^8 + \cdots$$

- Ereditarieta' dei logaritmi **analoga** a twist-2

$$\begin{aligned}
\mathcal{P}_1 &= 4\,S_1 \qquad \boxed{\sim \log N} \\
\mathcal{P}_2 &= -2\,S_3 \\
\mathcal{P}_3 &= 3\,S_5 - 2\,\Phi_{1,1,3} + \zeta_2 \cdot (-2\,S_3) \\
\mathcal{P}_4 &= 4\,S_1 \cdot \widehat{A}_4 + B_4 + 2\,\zeta_2 \cdot (3\,S_5 - 2\,\Phi_{1,1,3}) \\
& \boxed{\sim \frac{\log N}{N^2}}
\end{aligned}$$

- Forma generale del kernel RR

$$\begin{aligned}
\mathcal{P}(N) &= S_1 \cdot \left(\alpha_{\rm ph} + \widehat{A}\right) + B \\
\widehat{A} &= \mathcal{O}(1/N^2) \\
\widehat{A}, B &: \quad \textbf{senza logaritmi}
\end{aligned}$$

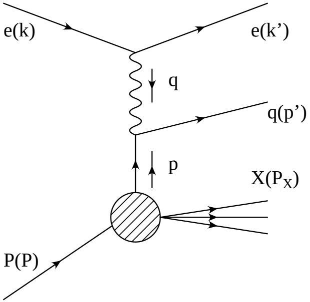
- Molto piu' semplice che nel caso delle γ

$$\gamma(N) = \alpha_{\rm ph} \log N + \sum_{k \geq 0} \frac{1}{N^k} \sum_{m=0}^{k} a_{k,m} \log^m N$$

- spirito del teorema LBK

DIS and splitting functions

The parton model

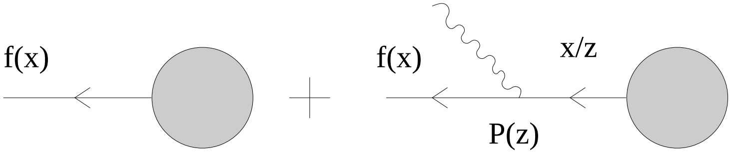


- The naive version of DIS

$$eP \to eX$$

is described in the parton model by distribution functions without evolution $f(x, Q^2) \equiv f(x)$

- The collinear emission is **absorbed** in the pdf's evolution



Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution equations

$$\frac{d}{dt} f_a^{\mathrm{NS}}(x,t) = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}\right) f_a^{\mathrm{NS}}(y,t).$$

$$f(x, t+dt) = f(x,t) + \frac{\alpha(t)}{2\pi} P \otimes F\, dt + \cdots ,$$

- The kernel $P(x)$ is a **splitting function**

Splitting functions and anomalous dimensions

- The DIS cross section involves the hadronic tensor

$$W(\nu, q^2) = \frac{1}{4\pi} \int d^4 z e^{-i q\cdot z} \langle P| J\left(\frac{z}{2}\right) J\left(-\frac{z}{2}\right) |P\rangle.$$

  which can be expanded on the light-cone in the DIS regime

$$J\left(\frac{z}{2}\right) J\left(-\frac{z}{2}\right) = \sum_{a,N} C_a^N(z^2)\, z^{\mu_1} \cdots z^{\mu_N} \boxed{\mathcal{O}^N_{\mu_1\ldots\mu_N,a}(0)}$$

- The splitting functions are related to anomalous dimensions of twist-2 operators with Lorentz spin N. **Schematically**

$$\int_0^1 dx\, x^{N-1}\, P(x) = -2\pi \boxed{\gamma_{\mathcal{O}}(N)}$$

Twist-3 $\mathfrak{sl}(2)$

- Extension of KLOV at twist-3, **4 loops** [Beccaria, 07]

$$\mathbb{O} = \mathrm{Tr}\left(D_+^{n_1}\varphi\, D_+^{n_2}\varphi\, D_+^{n_3}\varphi\right), \qquad n_1+n_2+n_3 = N$$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{s} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\, \frac{1-\dfrac{g^2}{x_k^+ x_j^-}}{1-\dfrac{g^2}{x_k^- x_j^+}}\, e^{2i\vartheta_{kj}},$$

- **Generalized KLOV principle**
- 4 loop results in closed form !

$$\begin{aligned}
\gamma_3^{(1)} &= 4\,S_1\,, \\
\gamma_3^{(2)} &= -2\,(S_3 + 2\,S_1\,S_2) \\
\gamma_3^{(3)} &= 5\,S_5 + 6\,S_2\,S_3 - 8\,S_{3,1,1} + 4\,S_{4,1} - 4\,S_{2,3} + \\
&\quad + S_1\,(4\,S_2^2 + 2\,S_4 + 8\,S_{3,1}),
\end{aligned}$$

$$\gamma_3^{(4)} = \gamma_3^{(4,\text{no dressing})} + \gamma_3^{(4,\text{dressing})}.$$

$$\gamma_3^{(4,\text{dressing})} = -8\,\beta\,S_1\,S_3.$$

$$\begin{aligned}
\gamma_{3,s}^{(4,\text{no dr.})} \quad = \quad & \frac{1}{2}\,S_7 + 7\,S_{1,6} + 15\,S_{2,5} - 5\,S_{3,4} - 29\,S_{4,3} - 21\,S_{5,2} - 5\,S_{6,1} \\
& -40\,S_{1,1,5} - 32\,S_{1,2,4} + 24\,S_{1,3,3} + 32\,S_{1,4,2} - 32\,S_{2,1,4} + 20\,S_{2,2,3} \\
& +40\,S_{2,3,2} + 4\,S_{2,4,1} + 24\,S_{3,1,3} + 44\,S_{3,2,2} + 24\,S_{3,3,1} + 36\,S_{4,1,2} + 36\,S_{4,2,1} \\
& +24\,S_{5,1,1} + 80\,S_{1,1,1,4} - 16\,S_{1,1,3,2} + 32\,S_{1,1,4,1} - 24\,S_{1,2,2,2} + 16\,S_{1,2,3,1} \\
& -24\,S_{1,3,1,2} - 24\,S_{1,3,2,1} - 24\,S_{1,4,1,1} - 24\,S_{2,1,2,2} + 16\,S_{2,1,3,1} - 24\,S_{2,2,1,2} \\
& -24\,S_{2,2,2,1} - 24\,S_{2,3,1,1} - 24\,S_{3,1,1,2} - 24\,S_{3,1,2,1} \\
& -24\,S_{3,2,1,1} - 24\,S_{4,1,1,1} - 64\,S_{1,1,1,3,1}.
\end{aligned}$$

where S_a are computed at $N/2$, the half-spin

- GL reciprocity **OK** $\surd$ [Beccaria, Marchesini, Dokshitzer, 07]
  cusp anomaly **OK** $\surd$

$\mathfrak{sl}(2|1)$ sector: universality of $\mathbb{O} = \mathrm{Tr}\, D_+^N(\lambda^3)$

- How can we extend the multi-loop analysis

$$\mathrm{Tr}\{D_+^N(\varphi^3)\} \quad \xrightarrow{?} \quad \mathrm{Tr}\{D_+^N(\lambda^3)\}$$

- The sector $\mathfrak{sl}(2|1) \supset \mathfrak{sl}(2)$ and contains three-fermion operators

$$\begin{aligned} \mathbb{O}(Z_1, \dots, Z_L) &= \mathrm{Tr} \prod_{i=1}^{L} \Phi_i(Z_i), \\ \Phi(Z) &= i\,\varphi(Z) + \theta\,\psi(Z), \qquad Z = (z, \theta). \end{aligned}$$

- SUSY universality [Beccaria, 07]

Theorem: $\gamma^{\lambda\lambda\lambda}(N) = \gamma_{\mathrm{twist}-2}(N+2)$

- GL reciprocity **OK** $\surd$ [Beccaria, 07]

Sector $\mathfrak{su}(1,2)$: 3-gluons $\mathbb{O} = \mathrm{Tr}\, D_+^N(A^3)$

A more difficult example in full glory

- At one loop we consider operators in light-cone gauge

$$\begin{aligned}\mathbb{O}_N &= \mathrm{Tr}\{\partial_+^{n_1} A\, \partial_+^{n_2} A\, \partial_+^{n_3} A\}, \\ N &= n_1 + n_2 + n_3\end{aligned}$$

- It is not a closed sector beyond 1 loop
- We must work in the full $\mathfrak{psu}(2,2|4)$ theory
- Identify the correct superconformal multiplet
- Decompose $V_F^{\otimes 3}$ where

$$\{\varphi, \quad , \lambda_\alpha, \quad \overline{\lambda}^{\dot\alpha}, \quad F_{\alpha\beta}, \quad \overline{F}_{\dot\alpha\dot\beta}\} \qquad \sim V_F$$

- General superconformal primary

$$\mathcal{V}^{\Delta,B}_{[j,\bar{j}][\lambda_1,\lambda_2,\lambda_3]},$$

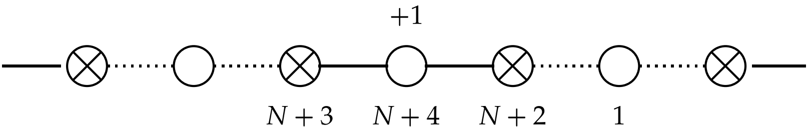
- The decomposition is [Beisert, 04]

$$(V_F \otimes V_F \otimes V_F)_S = \bigoplus_{\substack{n=0 \\ k\in\mathbb{Z}}}^{\infty} c_n \left[V_{2k,n} + V_{2k+1,n+3}\right],$$

- Three-gluon operators are descendants in the module

$$V_{2,N} = \mathcal{V}^{N+4,1}_{[\frac{N}{2}+1,\frac{N}{2}][0,0,0]}.$$

- Apply the long-range Bethe equations to the state with quantum numbers



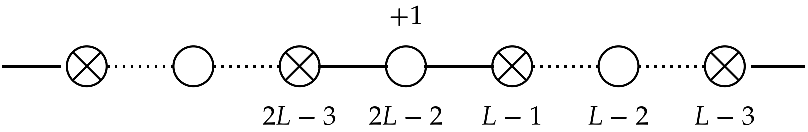
- **New generalization of the KLOV principle:** [Beccaria, 07]
  $n = \frac{N}{2} + 1$ e $S_{\mathbf{a}} \equiv S_{\mathbf{a}}(n)$

$$
\begin{aligned}
\gamma_1 &= 4\,S_1 + \frac{2}{n+1} + 4, \\
\gamma_2 &= -2\,S_3 - 4\,S_1\,S_2 - \frac{2\,S_2}{n+1} - \frac{2\,S_1}{(n+1)^2} - \frac{2}{(n+1)^3} + \\
&\quad -4\,S_2 - \frac{2}{(n+1)^2} - 8, \\
\gamma_3 &= 5\,S_5 + 6\,S_2\,S_3 - 4\,S_{2,3} + 4\,S_{4,1} - 8\,S_{3,1,1} \\
&\quad + \left(4\,S_2^2 + 2\,S_4 + 8\,S_{3,1}\right)\,S_1 \\
&\quad + \frac{-S_4 + 4\,S_{2,2} + 4\,S_{3,1}}{n+1} + \frac{4\,S_1\,S_2 + S_3}{(n+1)^2} + \frac{2\,S_1^2 + 3\,S_2}{(n+1)^3} \\
&\quad + \frac{6\,S_1}{(n+1)^4} + \frac{4}{(n+1)^5} - 2\,S_4 + 8\,S_{2,2} + 8\,S_{3,1} \\
&\quad + \frac{4\,S_2}{(n+1)^2} + \frac{4\,S_1}{(n+1)^3} + \frac{6}{(n+1)^4} + 8\,S_2 + 32,
\end{aligned}
$$

- Recently extended to 4 loops [Beccaria, Forini, 08]
- GL reciprocity **OK** √ [Beccaria, Forini, 08]
  cusp anomaly **OK** √

Closed formulae and Hyperintegrability ?

- Higher order gluon condensates $\mathrm{Tr}\, F^L$ [Beccaria, Forini, 07]



$$\gamma_L(g) = L\left(2 + 3\,g^2 + \sum_{n\geq 2} c_n(L)\,g^{2\,n}\right).$$

- Hypermagnets [Beccaria, Staudacher, Rej, Zieme, to appear]

$$\mathbb{O} = \mathrm{Tr}\,\{\varphi\,\varphi\,D^n\,\overline{D}^m\,\varphi\,\}$$

- Sum rules for higher twists in $\mathfrak{sl}(2)$ [Beccaria, Catino, 08]