Elliptic solutions of isentropic ideal compressible fluid flow in (3+1) dimensions

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The isentropic fluid dynamics equations

Euler equations

- We consider a system of equations describing a nonstationary ideal compressible fluid flow in $(3+1)$ dimensions,

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + (\vec{u} \cdot \nabla)\rho + \rho \, div \, \vec{u} = 0, \\ &\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right) + \nabla p = 0, \\ &\frac{\partial S}{\partial t} + (\vec{u} \cdot \nabla) S = 0. \end{aligned} \tag{1}$$

- Here ρ, p and S are the density, pressure and entropy respectively and $\vec{u} = (u^1, u^2, u^3)$ is a vector field describing the fluid velocity.

The isentropic fluid dynamics equations

The isentropic model

- System (1) can be reduced to the hyperbolic system describing an isentropic flow for which the state of the medium has the form

$$p = f(\rho, S).$$

- It becomes

$$\begin{aligned} D\rho + \rho\, div\, \vec{u} &= 0, \\ \rho D\vec{u} + \nabla p &= 0, \\ Dp + \rho a^2\, div\, \vec{u} &= 0, \end{aligned} \tag{2}$$

where $a^2 = f_\rho$ and D denotes the convective derivative,

$$D = \frac{\partial}{\partial t} + (\vec{u} \cdot \nabla).$$

The isentropic fluid dynamics equations

The isentropic model

- The isentropic model requires a^2 to be a function of the denstiy only, i.e.

$$\nabla p = a^2(\rho)\nabla\rho, \quad \frac{d\rho}{\rho} = \kappa\frac{da}{a}, \quad \kappa = \frac{2}{\gamma - 1},$$

  γ being the adiabatic exponent and $a = (\gamma p/\rho)^{1/2}$ stands for the velocity of sound in the medium.
- Under these assumptions, the system (1) can be reduced to a system of four equations in four unknowns,

$$\boxed{\begin{aligned} Da + \kappa^{-1} a\, div\, \vec{u} &= 0, \\ D\vec{u} + \kappa a \nabla a &= 0. \end{aligned}} \qquad (3)$$

The isentropic fluid dynamics equations - Matrix form

Notations

- Independent variables : $x = (t, x^1, x^2, x^3) \in X \subset \mathbb{R}^4$
- Dependent variables : $u = (a, \vec{u}) \in U \subset \mathbb{R}^4$
- Partial derivatives : $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$

Matrix form

Using these notations, the isentropic system reads in matrix evolutionary form

$$u_t + \sum_{i=1}^{3} A^i(u) u_i = 0,$$

with

$$A^i = \begin{pmatrix} u^i & \delta_{i1}\kappa^{-1}a & \delta_{i2}\kappa^{-1}a & \delta_{i3}\kappa-1a \\ \delta_{i1}\kappa a & u^i & 0 & 0 \\ \delta_{i2}\kappa a & 0 & u^i & 0 \\ \delta_{i3}\kappa a & 0 & 0 & u^i \end{pmatrix}, \quad i = 1, 2, 3. \tag{4}$$

The isentropic fluid dynamics equations

Classical symmetries

The largest Lie point symmetry algebra of the isentropic model (4) is the Galilean similitude algebra generated by the 12 differential operators

$$P_\mu = \partial_{x^\mu}, \quad J_k = \epsilon_{kij} \left(x^i \partial_{x^j} + u^i \partial_{u^j} \right), \quad K_i = t\partial_{x^i} + \partial_{u^i}, \quad i = 1, 2, 3,$$
$$F = t\partial_t + x^i \partial_{x^i}, \quad G = -t\partial_t + a\partial_a + u^i \partial_{u^i}.$$

When $\gamma = 5/3$, the algebra is generated by 13 differential operators, including the 12 operators (8) and a projective transformation

$$C = t(t\partial_t + x^i \partial_{x^i} - a\partial_a) + (x^i - tu^i)\partial_{u^i}.$$

The isentropic fluid dynamics equations

Dispersion relation

The admissible wave vectors $\lambda = (\lambda_0, \vec{\lambda})$ for the isentropic model are obtained from the dispersion relation

$$\det\left(\lambda_0 I_4 + \lambda_i(u) A^i(u)\right) = (\lambda_0 + \vec{u}\cdot\vec{\lambda})^2 \left[(\lambda_0 + \vec{u}\cdot\vec{\lambda})^2 - a^2|\lambda|^2\right] = 0. \quad (5)$$

Wave vectors

There are two types of wave vectors

- Potential : $\lambda^E = (\epsilon a + \vec{e}\cdot\vec{u}, -\vec{e}), \quad \epsilon = \pm 1,$
- Rotational : $\lambda^S = ([\vec{u}, \vec{e}, \vec{m}], -\vec{e}\times\vec{m}), \quad |\vec{e}|^2 = 1,$
  where $\vec{m}$ is an arbitrary vector and we denote $[\vec{u}, \vec{e}, \vec{m}] = \det(\vec{u}, \vec{e}, \vec{m})$.

The conditional symmetry method - Definitions

Invariance conditions

To any quasilinear hyperbolic homogeneous system of l partial differential equations of the first order

$$A^{\mu i}_{\ \ \alpha}(u) u^{\alpha}_{i} = 0, \quad \mu = 1, \ldots, l, \tag{6}$$

we associate the subvarieties

$$S_{\Delta} = \{(x, u^{(1)}) : A^{\mu i}_{\ \ \alpha}(u) u^{\alpha}_{i} = 0, \quad \mu = 1, \ldots, l\} \tag{7}$$

$$S_{Q} = \{(x, u^{(1)}) : \xi^{i}_{a}(u) u^{\alpha}_{i} = 0, \quad \alpha = 1, \ldots, q, \quad a = 1, \ldots, p-k\} \tag{8}$$

where the $p-k$ vectors ξ_a are orthogonal to the set of k linearly independent fixed wave vectors,

$$\xi^{i}_{a} \lambda^{A}_{i} = 0, \quad A = 1, \ldots, k, \quad a = 1, \ldots, p-k.$$

The constraints $\xi^{i}_{a}(u) u^{\alpha}_{i} = 0$ defining the subvariety S_Q are called the invariance conditions.

The conditional symmetry method - Definitions

Conditional symmetry

A vector field X_a is called a conditional symmetry of the original system (6) if X_a is tangent to $S = S_\Delta \cap S_Q$, i.e.

$$pr^{(1)} X_a \Big|_S \in T_{(x,u^{(1)})} S$$

where

$$pr^{(1)} X_a = X_a - \xi^i_{a,u^\beta} u^\beta_j u^\alpha_i \frac{\partial}{\partial u^\alpha_j}, \quad a = 1, \dots, p-k.$$

Conditional symmetry algebra

An Abelian Lie algebra L spanned by the vector fields $\{X_1, \dots, X_{p-k}\}$ is called a conditional symmetry algebra of the original system if the condition

$$pr^{(1)} X_a \left(A^i(u) u_i \right) \Big|_S = 0, \quad a = 1, \dots, p-k \tag{9}$$

is satisfied.

The conditional symmetry method - Proposition

A nondegenerate quasilinear hyperbolic system of first order PDEs

$$A^{\mu i}{}_{\alpha}(u) u_i^{\alpha} = 0, \quad \mu = 1, \ldots, l, \tag{10}$$

in p independent variables and q dependent variables admits a $(p-k)$-dimensional conditional symmetry algebra L if and only if there exist $(p-k)$ linearly independent vector fields

$$X_a = \xi_a^i(u) \frac{\partial}{\partial x^i}, \quad ker\left(A^i(u)\lambda_i^A\right) \neq 0, \quad \lambda_i^A \xi_a^i = 0, \tag{11}$$

$a = 1, \ldots, p-k$, $A = 1, \ldots, k$, ,which satisfy,on some neighborhood of (x_0, u_0) of S, the conditions

$$tr\left(A^{\mu} \frac{\partial f}{\partial r} \lambda\right) = 0, \quad \mu = 1, \ldots, l, \quad A = 1, \ldots, k, \quad s = 1, \ldots, k-1,$$

$$tr\left(A^{\mu} \frac{\partial f}{\partial r} \eta_{(a_1} \frac{\partial f}{\partial r} \ldots \eta_{a_s)} \frac{\partial f}{\partial r} \lambda\right) = 0, \quad \eta_{a_s} = \left(\frac{\partial \lambda_{a_s}^A}{\partial u^{\alpha}}\right) \in \mathbb{R}^{k \times q}.$$

The conditional symmetry method - Proposition

Solutions of the system (10) which are invariant under the Lie algebra L are precisely rank-k solutions defined implicitly by the following set of relations between the variables u^α, r^A and x^i

$$u = f\left(r^1(x,u), \ldots, r^k(x,u)\right), \quad r^A(x,u) = \lambda_i^A(u)x^i, \quad A = 1, \ldots, k, \tag{12}$$

for some function $f : \mathbb{R}^k \to \mathbb{R}^q$ and $rank(u_i^\alpha) = k$.

Isentropic flow with sound velocity depending on t only

The equations

- We consider here the isentropic flow of an ideal and compressible fluid in the case when the sound velocity depends on time t only.
- The system of equations (3) in $(k+1)$ dimensions becomes

$$\begin{aligned} & u_t + (u \cdot \nabla) u = 0, \\ & a_t + \kappa^{-1} a \, div \, u = 0, \\ & a_{x^j} = 0, \quad a > 0, \quad \kappa = 2(\gamma - 1)^{-1}, \quad j = 1, \ldots, k. \end{aligned} \tag{13}$$

- We show that the CSM approach enables us to construct general rank-k solutions.

Isentropic flow with sound velocity depending on t only

Change of coordinates

The change of coordinates on $\mathbb{R}^{k+1} \times \mathbb{R}^{k+1}$

$$\begin{aligned} &\bar{t} = t, \quad \bar{x}^1 = x^1 - u^1 t, \quad \dots, \quad \bar{x}^k = x^k - u^k t, \\ &\bar{a} = a, \quad \bar{u}^1 = u^1, \quad \dots, \bar{u}^k = u^k. \end{aligned} \tag{14}$$

transforms system (13) into

$$\frac{\partial \bar{\mathbf{u}}}{\partial \bar{t}} = 0, \quad \frac{\partial \bar{a}}{\partial \bar{t}} + \kappa^{-1} \bar{a} \operatorname{tr} \left(\left((\mathcal{I}_k + \bar{t} D\bar{\mathbf{u}}(\bar{\mathbf{x}}))^{-1} D\bar{\mathbf{u}}(\bar{\mathbf{x}}) \right) \right) = 0, \quad \frac{\partial \bar{a}}{\partial \bar{\mathbf{x}}} = 0, \tag{15}$$

where $\bar{\mathbf{u}} = (\bar{u}^1, \dots, \bar{u}^k)$, $D\bar{\mathbf{u}}(\bar{\mathbf{x}}) = \partial \bar{\mathbf{u}} / \partial \bar{\mathbf{x}} \in \mathbb{R}^{k \times k}$ is the Jacobian matrix and $\bar{\mathbf{x}} = (\bar{x}^1, \dots, \bar{x}^k) \in \mathbb{R}^k$.

Isentropic flow with sound velocity depending on t only

Rank-k solution

- The general solution of the conditions $\frac{\partial \bar{\mathbf{u}}}{\partial \bar{t}} = 0$ and $\frac{\partial \bar{a}}{\partial \bar{\mathbf{x}}} = 0$ is

$$\bar{\mathbf{u}}(\bar{t}, \bar{\mathbf{x}}) = f(\bar{\mathbf{x}}), \quad \bar{a}(\bar{t}, \bar{\mathbf{x}}) = \bar{a}(\bar{t}) > 0 \tag{16}$$

  for any function $f : \mathbb{R}^k \to \mathbb{R}^k$ and $\bar{a} : \mathbb{R} \to \mathbb{R}$.
- Making use of the Jacobi trace identity

$$\frac{\partial}{\partial \bar{t}} (\ln \det B) = \mathrm{tr} \left(B^{-1} \frac{\partial B}{\partial \bar{t}} \right), \tag{17}$$

we obtain from (15)

$$\frac{\partial}{\partial \bar{t}} \left[\ln \left(|\bar{a}(\bar{t})|^{\kappa} \det \left(\mathcal{I}_k + \bar{t} Df(\bar{\mathbf{x}}) \right) \right) \right] = 0. \tag{18}$$

Isentropic flow with sound velocity depending on t only

Rank-k solution

- This implies

$$\frac{\partial^2}{\partial \bar{\mathbf{x}} \partial \bar{t}} \left[\ln \left(\det \left(I_3 + \bar{t} Df(\bar{\mathbf{x}}) \right) \right) \right] = 0 \Rightarrow \det \left(I_k + \bar{t} Df(\bar{\mathbf{x}}) \right) = \alpha(\bar{\mathbf{x}}) \beta(\bar{t}) \tag{19}$$

  where α and β are arbitrary functions of their argument. Evaluation at $\bar{t} = 0$ implies $\alpha(\bar{\mathbf{x}}) = \beta(0)^{-1}$.

- Thus,

$$\det \left(I_k + \bar{t} Df(\bar{\mathbf{x}}) \right) = \frac{\beta(\bar{t})}{\beta(0)} \Rightarrow \frac{\partial}{\partial \bar{\mathbf{x}}} \det \left(I_k + \bar{t} Df(\bar{\mathbf{x}}) \right) = 0. \tag{20}$$

- Equation (20) is satisfied if and only if the coefficients $p_1, \ldots, p_n$ of the characteristic polynomial of the matrix $Df(\bar{\mathbf{x}})$ are constant.

Isentropic flow with sound velocity depending on t only

Rank-k solution

- The general rank-k solution of (13) is, in the original coordinates

$$u = f(x^1 - u^1 t, \ldots, x^k - u^k t), \quad a(t) = A_1(1 + p_1 t + \ldots + p_k t^k)^{-1/\kappa}, \tag{21}$$

with the Cauchy data

$$t = 0, \quad u(0, x^1, \ldots, x^k) = f(x^1, \ldots, x^k) \in \mathbb{R}^k, \quad a(0) = A_1 \in \mathbb{R}^+.$$

- Note also that the solution is invariant under the vector field

$$X = \frac{\partial}{\partial t} + \sum_{j=1}^{k} u^j \frac{\partial}{\partial x^j}.$$

Isentropic flow with sound velocity depending on t only

Rank-2 solution

- We illustrate here the results for $k = 2$.
- The rank-2 solution is invariant under the vector field

$$X = \frac{\partial}{\partial t} + u^1 \frac{\partial}{\partial x^1} + u^2 \frac{\partial}{\partial x^2}.$$

- The requirement that the coefficients p_n of the characteristic polynomial of the Jacobi matrix $Df(\bar{x})$ are constant means that

$$\det\left(Df(\bar{x})\right) = B_1, \quad \mathrm{tr}\left(Df(\bar{x})\right) = 2C_1, \quad B_1, C_1 \in \mathbb{R}, \tag{22}$$

where $B_1 = p_0$ and $2C_1 = p_1$.

Isentropic flow with sound velocity depending on t only

Rank-2 solution

The general rank-2 solution is then implicitly defined by

$$\begin{aligned} u^1(t,x,y) &= C_1(x^1 - u^1 t) + \frac{\partial h}{\partial r^2}(x^1 - u^1 t, x^2 - u^2 t), \\ u^2(t,x,y) &= C_1(x^2 - u^2 t) - \frac{\partial h}{\partial r^1}(x^1 - u^1 t, x^2 - u^2 t), \\ a(t) &= A_1((1 + C_1 t)^2 + B_1 t^2)^{-1/\kappa}, \quad A_1 \in \mathbb{R}^+, \end{aligned} \tag{23}$$

where the function h depends on two variables $r^1 = x^1 - u^1 t$ and $r^2 = x^2 - u^2 t$ and satisfies the nonhomogeneous Monge-Ampère (MA) equation

$$h_{r^1 r^1} h_{r^2 r^2} - h^2_{r^1 r^2} = b, \quad b = B_1 - C_1^2. \tag{24}$$

Isentropic flow with sound velocity depending on t only

Half-Legendre transformation

- According to Goursat, the following half-Legendre transformation $(r^1, r^2, h) \to (s, r^2, h)$,

$$\tilde{h}(z, r^2) = h(s, r^2) - s h_s(s, r^2) \tag{25}$$

  with

$$z = h_s(s, r^2), \quad h_{ss} \neq 0, \tag{26}$$

  allows us to transform the MA equation (24) into the linear Laplace-Beltrami equation

$$\tilde{h}_{r^2 r^2} + b \tilde{h}_{zz} = 0. \tag{27}$$

- Using the Goursat approach, we can associate to every solution $\tilde{h}(s, r^2)$ of the Laplace-Beltrami equation (27) a solution $h(r^1, r^2)$ of the Monge-Ampère equation (24).

Isentropic flow with sound velocity depending on t only

Solutions of the Monge-Ampère equations

Assuming that the constant b has been normalized to ± 1 we obtain,

$$b = -1: \qquad i)\ h = \left(\frac{r^1}{9} + \frac{2}{3}(r^2)^2\right)\left(-6r^1 - 36(r^2)^2\right)^{1/2}$$

$$ii)\ h = r^1\left(\ln\left(-\frac{r^1}{A_2 e^{2r^2} + B_2}\right) + r^2 - 1\right), \quad A_2, B_2 \in \mathbb{R}$$

$$b = 1: \qquad i)\ h = -\frac{(r^1)^2}{8} - 2(r^2)^2$$

$$ii)\ h = -\frac{1}{108}(72(r^2)^2 - 12r^1)\sqrt{36(r^2)^2 - 6r^1}$$

$$b = 0: \qquad i)\ h = -\frac{1}{4}(r^1 + r^2)^2$$

$$ii)\ h = r^1 \arcsin\left(\frac{r^1}{1 + r^2}\right) + (1 + r^2)\sqrt{1 - \frac{(r^1)^2}{(1 + r^2)^2}}$$

where $r^1 = x^1 - u^1 t, r^2 = x^2 - u^2 t$.

Isentropic flow with sound velocity depending on t only

Isentropic solutions

Every solution of the Monge-Ampère equation leads to a rank-2 solution of the isentropic model.

Hyperbolic case ($b = -1$)

No	Solutions	Comments
1.i	$u^1 = -\frac{-C_1 r^1 \sqrt{-6 r^1 - 36 (r^2)^2} + 12 r^2 r^1 + 72 (r^2)^3}{\sqrt{-6 r^1 - 36 (r^2)^2}}$ $u^2 = \frac{C_1 r^2 \sqrt{-6 r^1 - 36 (r^2)^2} + r^1 + 6 (r^2)^2}{\sqrt{-6 r^1 - 36 (r^2)^2}}$ $a = A_1 (1 + 2 C_1 t + (2 {C_1}^2 - 1) t^2)^{-1/\kappa}$	$C_1 \in \mathbb{R}$
1.ii	$u^1 = r^1 \left(C_1 - \frac{(A_2 e^{2r^2} + B_2)^2 + 2 A_2 r^1 e^{2r^2}}{(A_2 e^{2r^2} + B_2)(r^1 + (r^2 - 1)(A_2 e^{2r^2} + B_2))} \right)$ $u^2 = C_1 r^2 + \frac{r^1}{r^1 + (r^2 - 1)(A_2 e^{2r^2} + B_2)}$ $a = A_1 (1 + 2 C_1 t + (2 {C_1}^2 - 1) t^2)^{-1/\kappa}$	$A_1, A_2, B_2, C_1 \in \mathbb{R}$

$r^1 = x^1 - u^1 t, r^2 = x^2 - u^2 t.$

Isentropic flow with sound velocity depending on t only

Elliptic case ($b = 1$)

No	Solutions	Comments
2.i	$u^1 = \frac{C_1 x + ({C_1}^2+1)tx - 4y}{(C_1+1)t^2+2C_1 t+1}$ $u^2 = \frac{1}{4}\frac{4({C_1}^2+1)ty+4C_1 y+x}{(C_1+1)t^2+2C_1 t+1}$ $a = A_1(1+2C_1 t+(2{C_1}^2+1)t^2)^{-1/\kappa}$	$C_1 \in \mathbb{R}$
2.ii	$u^1 = C_1 r^1 + \frac{12r^2(r^1-6(r^2)^2)}{\sqrt{36(r^2)^2-6r^1}}$ $u^2 = C_1 r^2 + \frac{r^1-6(r^2)^2}{\sqrt{36(r^2)^2-6r^1}}$ $a = A_1(1+2C_1 t+(2{C_1}^2+1)t^2)^{-1/\kappa}$	$A_1, C_1 \in \mathbb{R}$

$r^1 = x^1 - u^1 t, r^2 = x^2 - u^2 t.$

Isentropic flow with sound velocity depending on t only

Parabolic case ($b = 0$)

No	Solutions	Comments
3.i	$u^1 = C_1 r^1 + \frac{1}{2}(r^1 + r^2)$ $u^2 = C_1 r^2 - \frac{1}{2}(r^1 + r^2)$ $a = A_1(1 + 2C_1 t(1 + C_1 t))^{-1/\kappa}$	$A_1, C_1 \in \mathbb{R}$
3.ii	$u^1 = C_1 r^1 + \sqrt{1 - \left(\frac{r^1}{1+r^2}\right)^2}$ $u^2 = C_1 r^2 - \arcsin\left(\frac{r^1}{1+r^2}\right)$ $a = A_1(1 + 2C_1 t(1 + C_1 t))^{-1/\kappa}$	$A_1, C_1 \in \mathbb{R}$

$r^1 = x^1 - u^1 t, r^2 = x^2 - u^2 t.$

Elliptic rank-2 and rank-3 solutions

Table of rank-2 solutions

The conditional symmetry approach has been applied to the general isentropic model (3) to produce rank-2 and rank-3 solutions.

No	Type	Vector Fields	Riemann Invariants	Solutions
1	E_1S_1	$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}$ $X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}$ $\beta_i = -(\vec{e}^2 \times \vec{m}^2)_i(a + \vec{e}^1 \cdot \vec{u}) + e_i^1[\vec{u}, \vec{e}^2, \vec{m}^2]$ $\sigma_j = -e_1^1(\vec{e}^2 \times \vec{m}^2)_j + e_j^1(\vec{e}^2 \times \vec{m}^2)_1, j = 2, 3$	$r^1 = ((1+k)\bar{a}_1(r^1) + C_2)t - \vec{e}^1 \cdot \vec{x}$ $r^2 = Ct - [\vec{x}, \vec{e}^2, \vec{m}^2], \quad [\vec{e}^1, \vec{e}^2, \vec{m}^2] = 0$ $C_2 = (C_1 e_1^1 - e_3^1)^{-1}$	$\bar{a} = \bar{a}_1(r^1) + a_0, \quad [\vec{u}_2, \vec{e}^2, \vec{m}^2] = C$ $\vec{u} = k\bar{a}_1(r^1) + \vec{u}_2(r^2), \quad \bar{u}_2^3(r^2) = C_1\bar{u}_2^1(r^2)$ $a_0, C, C_1, C_2 \in \mathbb{R}$
2*a*	S_1S_2	$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$ $X_2 = \frac{\partial}{\partial x^3}$	$r^1 = x^1 - u^1 t$ $r^2 = x^2 - u^2 t$	$\bar{a} = a_0, \quad \bar{u}^1 = -\phi_{r^2}, \quad \bar{u}^2 = \phi_{r^1},$ $\phi = \varphi(\alpha_1 r^1 + \alpha_2 r^2) + \beta_1 r^1 + \beta_2 r^2 + \gamma,$ $\bar{u}^3 = \bar{u}^3(r^1, r^2), \quad a_0, \alpha_i, \beta_i, \gamma \in \mathbb{R}, i = 1, 2,$
2*b*	S_1S_2	$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$ $X_2 = \frac{\partial}{\partial x^3}$	$r^1 = x^1 - u^1 t$ $r^2 = x^2 - u^2 t$	$\bar{a} = a_0, \quad \bar{u}^2 = \bar{u}^3 = g(x^1 - x^2), \quad a_0 \in \mathbb{R},$ $\bar{u}^1 = b(x^1 - tg(x^1 - x^2), x^2 - tg(x^1 - x^2))$
2*c*	S_1S_2	$X_2 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}$ $X_3 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}$ $\beta_j = \lambda_j^2[\vec{u}, \vec{e}^1, \vec{m}^1] - \lambda_j^1[\vec{u}, \vec{e}^2, \vec{m}^2]$ $\sigma_i = \lambda_1^1\lambda_i^2 - \lambda_i^1\lambda_1^2$	$r^1 = \left(C_1 + \frac{\lambda_1^1}{\lambda_1^2}C_2\right)t - \vec{\lambda}^1 \cdot \vec{x}$ $r^2 = \left(C_2 + \frac{\lambda_1^2}{\lambda_1^1}C_1 + G(r^1)\right)t - \vec{\lambda}^2 \cdot \vec{x}$ $\lambda_i^j = -(\vec{e}^j \times \vec{m}^j)_i$ $G(r^1) = \frac{1}{\lambda_1^1}\left((\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2)\bar{u}_1^2(r^1)\right.$ $\left. + (\lambda_1^1\lambda_3^2 - \lambda_1^2\lambda_3^1)\bar{u}_1^3(r^1)\right)$	$\bar{a} = a_0, \quad a_0, C_1, C_2 \in \mathbb{R}$ $\bar{u}^1 = \frac{1}{\lambda_1^1}(C_1 - \lambda_2^1\bar{u}_1^2(r^1) - \lambda_3^1\bar{u}_1^3(r^1))$ $- \left(\frac{\lambda_3^2}{\lambda_1^2}\eta + \frac{\lambda_2^2}{\lambda_1^2}\right)\bar{u}_2^2(r^2) + \frac{C_2}{\lambda_1^2}$ $\bar{u}^2 = \bar{u}_1^2(r^1) + \bar{u}_2^2(r^2)$ $\bar{u}^3 = \bar{u}_1^3(r^1) + \eta\bar{u}_2^2(r^2), \quad \eta = \frac{\lambda_1^2\lambda_2^1 - \lambda_1^1\lambda_2^2}{\lambda_1^1\lambda_3^2 - \lambda_3^1\lambda_1^2}$
3	$E_1E_2S_1$	$X = \frac{\partial}{\partial x^3} - \frac{\sigma_1}{\beta_{12}}\frac{\partial}{\partial t} + \frac{\beta_{23}}{\beta_{12}}\frac{\partial}{\partial x^1} + \frac{\beta_{31}}{\beta_{12}}\frac{\partial}{\partial x^2}$ $\sigma_1 = \epsilon_{ijk}\, e_i^1 e_j^2(\vec{e}^3 \times \vec{m})_k$ $\beta_{ij} = (e_j^1 e_i^2 - e_i^1 e_j^2)[\vec{u}, \vec{e}^3, \vec{m}^3]$ $+(e_j^2(\vec{e}^3 \times \vec{m}^3)_i - e_i^2(\vec{e}^3 \times \vec{m}^3)_j)(a + \vec{e}^1 \cdot \vec{u})$ $+(e_i^1(\vec{e}^3 \times \vec{m}^3)_j - e_j^1(\vec{e}^3 \times \vec{m}^3)_i)(a + \vec{e}^2 \cdot \vec{u})$	$r^1 = \frac{\beta\bar{u}_3^1(r^3)t - e_1^1 x^1 - e_2^1 x^2}{1 - \alpha(1+\kappa)t}$ $r^2 = \frac{-\beta\bar{u}_3^1(r^3)t - e_1^2 x^1 - e_2^2 x^2}{1 - \alpha(1+\kappa)t}$ $r^3 = x^3 - u_0^3 t$ $\beta = (1 + \kappa^{-1})/(e_1^1 - e_1^2)$	$\bar{a} = \frac{\alpha((e_1^1 + e_1^2)x^1 + (e_2^1 + e_2^2)x^2)}{1 - \alpha(1+\kappa)t}, \quad \bar{u}^3 = u_0^3$ $\bar{u}^1 = \frac{-\kappa\alpha\left(((e_1^1)^2 + (e_1^2)^2)x^1 + (e_1^1 e_2^1 + e_1^2 e_2^2)x^2\right) - \bar{u}_3^1(r^3)}{1 - \alpha(1+\kappa)t}$ $\bar{u}^2 = \kappa\alpha\left(\frac{e_2^1\left(\beta\bar{u}_3^1(r^3)t - e_1^1 x^1 - e_2^1 x^2\right)}{1 - \alpha(1+\kappa)t}\right.$ $\left. + \frac{e_2^2\left(-\beta\bar{u}_3^1(r^3)t - e_1^2 x^1 - e_2^2 x^2\right)}{1 - \alpha(1+\kappa)t}\right) + \frac{e_2^2 - e_2^1}{e_1^2 - e_1^1}\bar{u}_3^1(r^3)$ $\alpha, u_0^3 \in \mathbb{R}$

Table of rank-3 solutions

No	Type	Vector Fields	Riemann Invariants	Solutions
1	$E_1E_2E_3$	$X_1 = \frac{\partial}{\partial x^3} + \frac{\sigma_1}{\beta_3}\frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3}\frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3}\frac{\partial}{\partial x^2}$ $\sigma_1 = -[\vec{e}^1, \vec{e}^2, \vec{e}^3]$ $\beta_i = (\vec{e}^2 \times \vec{e}^3)_i(a + \vec{e}^1 \cdot \vec{u})$ $+(\vec{e}^1 \times \vec{e}^3)_i(a + \vec{e}^2 \cdot \vec{u})$ $+(\vec{e}^1 \times \vec{e}^2)_i(a + \vec{e}^3 \cdot \vec{u})$	$r^i = (1+\kappa)a_i(r^i)t - \vec{e}^i \cdot \vec{x},\ i = 1,2,3$ $\vec{e}^i \cdot \vec{e}^j = -1/\kappa,\ i \neq j = 1,2,3$	$\bar{a} = \bar{a}_1(r^1) + \bar{a}_2(r^2) + \bar{a}_3(r^3)$ $\vec{u} = \kappa(\vec{e}^1\bar{a}_1(r^1) + \vec{e}^2\bar{a}_2(r^2) + \vec{e}^3\bar{a}_3(r^3))$
$2a$	$E_1S_1S_2$	$X = e_1^2\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}$	$r^1 = ((1+k^{-1})f(r^1) + a_0 + u_0^3)t - x^3$ $r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3)$ $\frac{\partial r^3}{\partial t} + (f(r^1) + u_0^3)\frac{\partial r^3}{\partial x^3} = 0$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin g(r^2, r^3)$ $\bar{u}^2 = -\cos g(r^2, r^3), \quad \bar{u}^3 = f(r^1) + u_0^3$ $a_0, u_0^3 \in \mathbb{R}$
$2b$	$E_1S_1S_2$	$X = e_1^2\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}$	$r^1 = \frac{((1+k^{-1})B + a_0 + u_0^3)t - x^3}{1-(1+k^{-1})At}$ $r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3)$ $r^3 = \Psi\left[\frac{1}{A}(A(ka_0 - u_0^3)t + x^3 - ka_0 - B)((1+k)At - k)^{-k/k+1}\right]$	$\bar{a} = k^{-1}(Ar^1 + B) + a_0,$ $\bar{u}^1 = \sin g(r^2, r^3), \bar{u}^2 = -\cos g(r^2, r^3)$ $\bar{u}^3 = Ar^1 + B + u_0^3, \quad a_0, u_0^3 \in \mathbb{R}$
$2c$	$E_1S_1S_2$	$X = \frac{\partial}{\partial x^3}$	$r^1 = (k^{-1}f(r^1) + a_0)t - x^1 \cos f(r^1) - x^2 \sin f(r^1)$ $r^2 = -t\cos f(r^1) - x^2$ $r^3 = -t \sin f(r^1) + x^1$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin f(r^1)$ $\bar{u}^2 = -\cos f(r^1), \quad a_0 \in \mathbb{R}$ $\bar{u}^3 = g(r^2 \cos f(r^1) + r^3 \sin f(r^1))$

Elliptic rank-2 and rank-3 solutions

Rank-2 and rank-3 solutions

- Several of the obtained solutions possess a certain amount of freedom since they depend on arbitrary variables of one or two Riemann invariants.
- These arbitrary functions allow us to change the geometrical properties of the fluid flow in such a way as to exclude the presence of singularities.
- To construct bounded solutions of soliton-type expressed in terms of elliptic functions, we submit the arbitrary functions appearing in the general solutions, say v, to the differential constraint in the form of the v^6-field Klein-Gordon equation in three independent variables

$$\Box_{(r^1,r^2,r^3)} v = c v^5, \quad c \in \mathbb{R}, \tag{28}$$

which is known to possess rich families of soliton-like solutions.

Reduction of the Klein-Gordon equation in Minkowski space $M(1,2)$ to a second order ODE

- The solution is of the form $v = \alpha(r)F(\xi), \xi = h(r), r = (r^1, r^2, r^3)$.
- We use the following basis for the Lie algebra $sim(1,2)$:
  - Dilation : $D = r^i \partial_{r^i} - \frac{1}{2} v \partial_v$
  - Translations : $P_a = \partial_{r^a}$
  - Rotations : $L_{ab} = r^a \partial_{r^b} - r^b \partial_{r^a}$
  - Lorentz boosts : $K_{1a} = -r^1 \partial_{r^a} - r^a \partial r^1$

  for $a \neq b = 1, 2, 3$.

No	Algebra	α	ξ	ODE
1	D, P_1	$\{4c[(r^2)^2 + (r^3)^2]\}^{-1/4}$	$\frac{1}{2} \arctan \frac{r^3}{r^2}$	$F'' + F + F^5 = 0$
2	D, L_{31}	$\{-c(r^1)^2/4\}^{-1/4}$	$\frac{(r^2)^2+(r^3)^2}{(r^1)^2}$	$\xi(1+\xi)F'' + \left(2\xi + \frac{3}{2}\right) F' + \frac{3}{16} F + F^5 = 0$
3	$D + \frac{1+q}{q} K_{12}, L_{23}$	$\{-\frac{(2q+1)}{c}\}^{1/4}(r^1 + r^2)^{q/2}$	$[(r^1)^2 - (r^2)^2 - (r^3)^2](r^1 + r^2)^q$	$F'' + \frac{3q+k}{2q+1} \frac{1}{\xi} F' + F^5 = 0, \quad q = -k/3, k-2, 4-3k$
4	$D + \frac{1}{2} K_{12}, L_1 - K_{13}$	$(9/4C)^{1/4}\{r^2 - (r^1 + r^3)^2/4\}^{-1/2}$	$\frac{6(r^3-r^1)+6r^2(r^1+r^3)-(r^1+r^3)^3}{8(r^2-(r^1+r^3)^2/4)^{3/2}}$	$(1+\xi^2)F'' + \frac{7}{3} F' + \frac{1}{3} F + F^5 = 0$

Reduction of the Klein-Gordon equation

The parity invariance of the Klein-Gordon equation suggests the substitution

$$F(\xi) = [H(\xi)]^{1/2}$$

which transforms the reduced equations into

$$H'' = \frac{H'^2}{2H} - 2(H + H^3), \tag{29}$$

$$H'' = \frac{H'^2}{2H} - \frac{1}{\xi(1+\xi)}\left[\left(2\xi + \frac{3}{2}\right)H' + \frac{3}{8}H + 2H^3\right], \tag{30}$$

$$H'' = \frac{H'^2}{2H} - \left[\frac{3q+k}{2q+1}\frac{1}{\xi}H' + 2H^3\right],\ a = \frac{3q+k}{2q+1} = (0, 4/3, 2), \tag{31}$$

$$H'' = \frac{H'^2}{2H} - \frac{1}{1+\xi^2}\left[\frac{7}{3}\xi H' + \frac{2}{3}H + 2H^3\right]. \tag{32}$$

A first integral

- Each of these equations admits a first integral

$$K' = \frac{1}{4} G g^2 \frac{(gH)'^2}{gH} - \frac{c_4}{4} (gH)^3 - 3 e_0 g H \tag{33}$$

in which the four sets of functions G, g and constants e_0, c_4 obey the respective conditions

$$G = -\frac{3c_4}{4},\ g^2 = \frac{4e_0}{c_4}, \tag{34}$$

$$G = -\frac{3c_4}{4} \xi(\xi + 1),\ g^2 = -\frac{64 e_0}{c_4} \xi, \tag{35}$$

$$G = -\frac{3c_4}{4}, \tag{36}$$

$$(a, e_0, g^2) = (0, 0, k_1),\ (4/3, 0, k_1 \xi^{4/3}),\ (2, e_0, -\frac{16 e_0}{c_4} \xi^2),$$

$$G = -\frac{3c_4}{4} (\xi^2 + 1),\ g = k_1 (1 + \xi^2)^{1/3},\ e_0 = 0, \tag{37}$$

A first integral

Under a transformation $(H, \xi) \to (U, \zeta)$ which preserves the Painlevé property,

$$H(\xi) = U(\zeta)/g(\xi), \quad \left(\frac{d\zeta}{d\xi}\right)^2 = \frac{1}{Gg^2} \tag{38}$$

the first integral (33) becomes autonomous

$$U'^2 - c_4 U^4 - 12 e_0 U^2 - 4K'U = 0, \quad c_4 \neq 0. \tag{39}$$

Elliptic solutions

- When $K' = 0$, U^{-1} is either a *sin*, *cos*, *sinh* or *cosh*, depending on the signs of the constants.
- When $K' \neq 0$, we integrate the first integral (39) in terms of the **Weierstrass** $\wp$-function using the following ansatz,

$$U(\zeta) = \frac{K'}{\wp(\zeta) - e_0}, \quad g_2 = 12 e_0^2, \quad g_3 = -8 e_0^3 - c_4 {K'}^2, \qquad (40)$$
$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2 \wp - g_3.$$

- One can then use Halphen's symmetric notation to express the obtained solution in terms of the **Jacobi elliptic** functions. The connection is given by

$$\frac{cs(z)}{h_1(u)} = \frac{ds(z)}{h_2(u)} = \frac{ns(z)}{h_3(u)} = \frac{u}{z} = \frac{1}{\sqrt{e_1 - e_3}}, \qquad (41)$$

where $h_\alpha(u) = \sqrt{\wp(u) - e_\alpha}$, $\alpha = 1, 2, 3$.

Elliptic solutions

Using appropriate normalizations for the constants e_0, c_4, K' and k_1, we obtain the general solutions of the reduced equations (29)-(32) in terms of the $\wp$-Weierstrass function.

- (29) : $e_0 = -1/3, c_4 = -4/3, K' = C$

$$F^2(\xi) = \frac{C}{\wp(\xi) + 1/3},\ g_2 = \frac{4}{3},\ g_3 = \frac{8}{27} + \frac{4}{3}C^2,\ C \in \mathbb{R}. \tag{42}$$

- (30) : $e_0 = k_0^{-2}/48, c_4 = -(4/3)k_0^{-2}, K' = C$

$$\begin{aligned} F^2(\xi) &= \frac{C\xi^{-1/2}}{\wp(\zeta) - \frac{1}{48k_0^2}},\ \zeta = -2k_0 \,\mathrm{Argth}\, \sqrt{\xi + 1}, \\ g_2 &= \frac{1}{192k_0^4},\ g_3 = -\frac{1}{13824k_0^6} + \frac{4C^2}{3k_0^2}, \end{aligned} \tag{43}$$

Elliptic solutions

- Similarly for the three cases of equation (31).

$$q = -k/3 : F^2(\xi) = \frac{C}{\wp(\xi)},\ g_2 = 0,\ g_3 = \frac{4C^2}{3},$$

$$q = 4 - 3k : F^2(\xi) = \frac{C\xi^{-2/3}}{\wp(\zeta)},\ \zeta = 3k_0\xi^{1/3},\ g_2 = 0,\ g_3 = \frac{4C^2}{3k_0^2},$$

$$q = k - 2 : F^2(\xi) = \frac{C\xi^{-1}}{\wp(\zeta) - \frac{1}{12k_0^2}},\ \zeta = k_0 \log \xi,$$

$$g_2 = \frac{1}{12k_0^4},\ g_3 = -\frac{1}{216k_0^6} + \frac{4C^2}{3k_0^2}.$$

Elliptic solutions

- Finally, we obtain for equation (32)

$$F^2(\xi) = \frac{C(\xi^2+1)^{-1/3}}{\wp(\zeta)},$$

$$\zeta = \xi \ {}_2F_1\left(\frac{1}{2}, \frac{5}{6}; \frac{3}{2}; -\xi^2\right), \quad g_2 = 0, \quad g_3 = \frac{4C^2}{3k_0^2},$$

- To construct bounded solutions, we submit arbitrary functions in every rank-2 and rank-3 solution obtained with the conditional symmetry method and select only solutions expressed in terms of the Weierstrass $\wp$-function.

Example of bounded rank-3 solution for the case $E_1 E_2 E_3$

We consider the case of a superposition of three potential waves $\lambda^{E_1}, \lambda^{E_2}, \lambda^{E_3}$ that intersect at a prescribed angle given by

$$\cos\phi_{ij} = -\frac{1}{\kappa}, \quad i \neq j = 1,2,3, \quad \kappa = \frac{2}{\gamma - 1}, \tag{44}$$

where ϕ_{ij} denotes the angle between the wave vectors $\vec{\lambda}^i$ and $\vec{\lambda}^j$. The solution reads in this case

$$\begin{aligned} \bar{a} &= \sum_{i=1}^{3} \bar{a}_i(r^i), \quad \vec{u} = \kappa \sum_{i=1}^{3} \bar{a}_i(r^i)\vec{e}^i, \\ r^i &= (1+\kappa)\bar{a}_i(r^i)t - \vec{e}^i \cdot \vec{x} \end{aligned} \tag{45}$$

Example of bounded rank-3 solution for the case $E_1 E_2 E_3$

Requiring that each function $\bar{a}_i(r^i)$ satisfies the first equation obtained by reducing the Klein-Gordon equation with the subgroup generated by D, P_1, we obtain the bounded rank-3 solution

$$\begin{aligned}
a &= \sum_{i=1}^{3} \frac{C_i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3} C_i^4) + \frac{1}{3}\right)^{1/2}}, \\
\vec{u} &= \kappa \sum_{i=1}^{3} \frac{C_i \vec{\lambda}^i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3} C_i^4) + \frac{1}{3}\right)^{1/2}}, \\
r^i &= -(1+\kappa) \frac{C_i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3} C_i^4) + \frac{1}{3}\right)^{1/2}} t + \vec{\lambda}^i \cdot \vec{x}, \quad i = 1, 2, 3.
\end{aligned} \tag{46}$$

This solution is interesting since it remains bounded for every value of the Riemann invariants r^i and represents a bounded solution with periodic flow velocities.

Table of rank-3 elliptic solutions for the case $E_1 E_2 E_3$

no	Riemann invariants	Solution	Type and comments
1	$r^i = -(1+\kappa)\dfrac{C_i}{\left(\wp\left(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4\right) + \frac{1}{3}\right)^{1/2}} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3} \dfrac{C_i}{\left(\wp\left(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4\right) + \frac{1}{3}\right)^{1/2}}$	Periodic solution
		$\vec{u} = \kappa \sum_{i=1}^{3} \dfrac{C_i \vec{\lambda}^i}{\left(\wp\left(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4\right) + \frac{1}{3}\right)^{1/2}}$	$C_i \in \mathbb{R}$
2	$r^i = -(1+\kappa)\left((r^i)^{-1/2} \dfrac{C_i}{\wp(\zeta_i, 12e_0^2, -8e_0^3 + 64C_i^2 e_0) - e_0}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3} \left((r^i)^{-1/2} \dfrac{C_i}{\wp(\zeta_i, 12e_0^2, -8e_0^3 + 64C_i^2 e_0) - e_0}\right)^{1/2}$	
	$\zeta_i = -2k_0 \text{arctanh}\sqrt{r^i + 1}$	$\vec{u} = \kappa \left(\sum_{i=1}^{3} \left((r^i)^{-1/2} \dfrac{C_i}{\wp(\zeta_i, 12e_0^2, -8e_0^3 + 64C_i^2 e_0) - e_0}\right)^{1/2} \vec{\lambda}^i\right)$	$e_0, \in \mathbb{R}, C_i > 0$
3a	$r^i = -(1+\kappa)\left(\dfrac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3} \left(\dfrac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)}\right)^{1/2}, \vec{u} = \kappa \sum_{i=1}^{3} \left(\dfrac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)}\right)^{1/2} \vec{\lambda}^i$	Periodic Solution
			$C_i > 0$
3b	$r^i = -(1+\kappa)\left(\dfrac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3} \left(\dfrac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})}\right)^{1/2}, \vec{u} = \kappa \sum_{i=1}^{3} \left(\dfrac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})}\right)^{1/2} \vec{\lambda}^i$	Bump
	$\zeta_i = 3k_0 (r^i)^{1/3}$		$k_0 \in \mathbb{R}, C_i > 0$
3c	$r^i = -(1+\kappa)\left(\dfrac{C_i (r^i)^{-1}}{(\wp(\zeta_i, 12e_0^2, -8e_0^3 + 16C_i^2 e_0) - e_0)}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3} \left(\dfrac{C_i (r^i)^{-1}}{(\wp(\zeta_i, 12e_0^2, -8e_0^3 + 16C_i^2 e_0) - e_0)}\right)^{1/2}$	Bump
	$\zeta_i = k_0 \ln r^i$	$\vec{u} = \sum_{i=1}^{3} \kappa \left(\dfrac{C_i (r^i)^{-1}}{(\wp(\zeta_i, 12e_0^2, -8e_0^3 + 16C_i^2 e_0) - e_0)}\right)^{1/2} \vec{\lambda}^i$	$e_0, \in \mathbb{R}, C_i > 0$
4	$r^i = -(1+\kappa)\left(\dfrac{C_i ((r^i)^2 + 1)^{-1/3}}{\wp\left(\zeta_i, 0, \frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3} \left(\dfrac{C_i ((r^i)^2 + 1)^{-1/3}}{\wp\left(\zeta_i, 0, \frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2}$	Anti-bump
	$\zeta_i = r^i \; {}_2F_1\left(\frac{1}{2}, \frac{5}{6}; \frac{3}{2}; -(r^i)^2\right)$	$\vec{u} = \kappa \sum_{i=1}^{3} \left(\dfrac{C_i ((r^i)^2 + 1)^{-1/3}}{\wp\left(\zeta_i, 0, \frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2} \vec{\lambda}^i$	$k_0, \in \mathbb{R}, C_i > 0$

Table of rank-2 solutions

No	Type	Vector Fields	Riemann Invariants	Solutions
1	E_1S_1	$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}$	$r^1 = ((1+k)\bar{a}_1(r^1) + C_2)t - \vec{e}^{\,1}\cdot\vec{x}$	$\bar{a} = \bar{a}_1(r^1) + a_0, \quad [\vec{u}_2, \vec{e}^{\,2}, \vec{m}^2] = C$
		$X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}$	$r^2 = Ct - [\vec{x}, \vec{e}^{\,2}, \vec{m}^2], \quad [\vec{e}^{\,1}, \vec{e}^{\,2}, \vec{m}^2] = 0$	$\vec{u} = k\bar{a}_1(r^1) + \vec{u}_2(r^2), \quad \bar{u}_2^3(r^2) = C_1\bar{u}_2^1(r^2)$
		$\beta_i = -(\vec{e}^{\,2}\times\vec{m}^2)_i(a + \vec{e}^{\,1}\cdot\vec{u}) + e_i^1[\vec{u}, \vec{e}^{\,2}, \vec{m}^2]$	$C_2 = (C_1e_1^1 - e_3^1)^{-1}$	$a_0, C, C_1, C_2 \in \mathbb{R}$
		$\sigma_j = -e_1^1(\vec{e}^{\,2}\times\vec{m}^2)_j + e_j^1(\vec{e}^{\,2}\times\vec{m}^2)_1, j = 2, 3$		
$2a$	S_1S_2	$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$	$r^1 = x^1 - u^1t$	$\bar{a} = a_0, \quad \bar{u}^1 = -\phi_{r^2}, \quad \bar{u}^2 = \phi_{r^1},$
		$X_2 = \frac{\partial}{\partial x^3}$	$r^2 = x^2 - u^2t$	$\phi = \varphi(\alpha_1r^1 + \alpha_2r^2) + \beta_1r^1 + \beta_2r^2 + \gamma,$
				$\bar{u}^3 = \bar{u}^3(r^1, r^2), \quad a_0, \alpha_i, \beta_i, \gamma \in \mathbb{R}, i = 1, 2,$
$2b$	S_1S_2	$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$	$r^1 = x^1 - u^1t$	$\bar{a} = a_0, \quad \bar{u}^2 = \bar{u}^3 = g(x^1 - x^2), \quad a_0 \in \mathbb{R},$
		$X_2 = \frac{\partial}{\partial x^3}$	$r^2 = x^2 - u^2t$	$\bar{u}^1 = b(x^1 - tg(x^1 - x^2), x^2 - tg(x^1 - x^2))$
$2c$	S_1S_2	$X_2 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}$	$r^1 = \left(C_1 + \frac{\lambda_1^1}{\lambda_1^2}C_2\right)t - \vec{\lambda}^1\cdot\vec{x}$	$\bar{a} = a_0, \quad a_0, C_1, C_2 \in \mathbb{R}$
		$X_3 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}$	$r^2 = \left(C_2 + \frac{\lambda_1^2}{\lambda_1^1}C_1 + G(r^1)\right)t - \vec{\lambda}^2\cdot\vec{x}$	$\bar{u}^1 = \frac{1}{\lambda_1^1}(C_1 - \lambda_2^1\bar{u}_1^2(r^1) - \lambda_3^1\bar{u}_1^3(r^1))$
		$\beta_j = \lambda_j^2[\vec{u}, \vec{e}^{\,1}, \vec{m}^1] - \lambda_j^1[\vec{u}, \vec{e}^{\,2}, \vec{m}^2]$	$\lambda_i^j = -(\vec{e}^{\,j}\times\vec{m}^j)_i$	$-\left(\frac{\lambda_3^2}{\lambda_1^2}\eta + \frac{\lambda_2^2}{\lambda_1^2}\right)\bar{u}_2^2(r^2) + \frac{C_2}{\lambda_1^2}$
		$\sigma_i = \lambda_1^1\lambda_i^2 - \lambda_i^1\lambda_1^2$	$G(r^1) = \frac{1}{\lambda_1^1}\left((\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2)\bar{u}_1^2(r^1)\right.$	$\bar{u}^2 = \bar{u}_1^2(r^1) + \bar{u}_2^2(r^2)$
			$\left.+ (\lambda_1^1\lambda_3^2 - \lambda_1^2\lambda_3^1)\bar{u}_1^3(r^1)\right)$	$\bar{u}^3 = \bar{u}_1^3(r^1) + \eta\bar{u}_2^2(r^2), \quad \eta = \frac{\lambda_1^2\lambda_2^1 - \lambda_1^1\lambda_2^2}{\lambda_1^1\lambda_3^2 - \lambda_3^1\lambda_1^2}$
3	$E_1E_2S_1$	$X = \frac{\partial}{\partial x^3} - \frac{\sigma_1}{\beta_{12}}\frac{\partial}{\partial t} + \frac{\beta_{23}}{\beta_{12}}\frac{\partial}{\partial x^1} + \frac{\beta_{31}}{\beta_{12}}\frac{\partial}{\partial x^2}$	$r^1 = \frac{\beta\bar{u}_3^1(r^3)t - e_1^1x^1 - e_2^1x^2}{1 - \alpha(1+\kappa)t}$	$\bar{a} = \frac{\alpha((e_1^1 + e_1^2)x^1 + (e_2^1 + e_2^2)x^2)}{1 - \alpha(1+\kappa)t}, \quad \bar{u}^3 = u_0^3$
		$\sigma_1 = \epsilon_{ijk}\,e_i^1e_j^2(\vec{e}^{\,3}\times\vec{m})_k$	$r^2 = \frac{-\beta\bar{u}_3^1(r^3)t - e_1^2x^1 - e_2^2x^2}{1 - \alpha(1+\kappa)t}$	$\bar{u}^1 = \frac{-\kappa\alpha\left(((e_1^1)^2 + (e_1^2)^2)x^1 + (e_1^1e_2^1 + e_1^2e_2^2)x^2\right) - \bar{u}_3^1(r^3)}{1 - \alpha(1+\kappa)t}$
		$\beta_{ij} = (e_j^1e_i^2 - e_i^1e_j^2)[\vec{u}, \vec{e}^{\,3}, \vec{m}^3]$	$r^3 = x^3 - u_0^3t$	$\bar{u}^2 = \kappa\alpha\left(\frac{e_2^1\left(\beta\bar{u}_3^1(r^3)t - e_1^1x^1 - e_2^1x^2\right)}{1 - \alpha(1+\kappa)t}\right.$
		$+(e_j^2(\vec{e}^{\,3}\times\vec{m}^3)_i - e_i^2(\vec{e}^{\,3}\times\vec{m}^3)_j)(a + \vec{e}^{\,1}\cdot\vec{u})$	$\beta = (1 + \kappa^{-1})/(e_1^1 - e_1^2)$	$\left.+\frac{e_2^2\left(-\beta\bar{u}_3^1(r^3)t - e_1^2x^1 - e_2^2x^2\right)}{1 - \alpha(1+\kappa)t}\right) + \frac{e_2^2 - e_2^1}{e_1^2 - e_1^1}\bar{u}_3^1(r^3)$
		$+(e_i^1(\vec{e}^{\,3}\times\vec{m}^3)_j - e_j^1(\vec{e}^{\,3}\times\vec{m}^3)_i)(a + \vec{e}^{\,2}\cdot\vec{u})$		$\alpha, u_0^3 \in \mathbb{R}$

Table of rank-3 solutions

No	Type	Vector Fields	Riemann Invariants	Solutions
1	$E_1E_2E_3$	$X_1 = \frac{\partial}{\partial x^3} + \frac{\sigma_1}{\beta_3}\frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3}\frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3}\frac{\partial}{\partial x^2}$	$r^i = (1+\kappa)a_i(r^i)t - \vec{e}^i \cdot \vec{x},\ i = 1,2,3$	$\bar{a} = \bar{a}_1(r^1) + \bar{a}_2(r^2) + \bar{a}_3(r^3)$
		$\sigma_1 = -[\vec{e}^1, \vec{e}^2, \vec{e}^3]$	$\vec{e}^i \cdot \vec{e}^j = -1/\kappa,\ i \neq j = 1,2,3$	$\vec{u} = \kappa(\vec{e}^1\bar{a}_1(r^1) + \vec{e}^2\bar{a}_2(r^2) + \vec{e}^3\bar{a}_3(r^3))$
		$\beta_i = (\vec{e}^2 \times \vec{e}^3)_i(a + \vec{e}^1 \cdot \vec{u})$		
		$+(\vec{e}^1 \times \vec{e}^3)_i(a + \vec{e}^2 \cdot \vec{u})$		
		$+(\vec{e}^1 \times \vec{e}^2)_i(a + \vec{e}^3 \cdot \vec{u})$		
$2a$	$E_1S_1S_2$	$X = e_1^2\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}$	$r^1 = ((1+k^{-1})f(r^1) + a_0 + u_0^3)t - x^3$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin g(r^2, r^3)$
			$r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3)$	$\bar{u}^2 = -\cos g(r^2, r^3), \quad \bar{u}^3 = f(r^1) + u_0^3$
			$\frac{\partial r^3}{\partial t} + (f(r^1) + u_0^3)\frac{\partial r^3}{\partial x^3} = 0$	$a_0, u_0^3 \in \mathbb{R}$
$2b$	$E_1S_1S_2$	$X = e_1^2\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}$	$r^1 = \frac{((1+k^{-1})B + a_0 + u_0^3)t - x^3}{1-(1+k^{-1})At}$	$\bar{a} = k^{-1}(Ar^1 + B) + a_0,$
			$r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3)$	$\bar{u}^1 = \sin g(r^2, r^3), \bar{u}^2 = -\cos g(r^2, r^3)$
			$r^3 = \Psi\left[\frac{1}{A}(A(ka_0 - u_0^3)t + x^3 - ka_0 - B)((1+k)At - k)^{-k/k+1}\right]$	$\bar{u}^3 = Ar^1 + B + u_0^3, \quad a_0, u_0^3 \in \mathbb{R}$
$2c$	$E_1S_1S_2$	$X = \frac{\partial}{\partial x^3}$	$r^1 = (k^{-1}f(r^1) + a_0)t - x^1 \cos f(r^1) - x^2 \sin f(r^1)$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin f(r^1)$
			$r^2 = -t\cos f(r^1) - x^2$	$\bar{u}^2 = -\cos f(r^1), \quad a_0 \in \mathbb{R}$
			$r^3 = -t \sin f(r^1) + x^1$	$\bar{u}^3 = g(r^2 \cos f(r^1) + r^3 \sin f(r^1))$

Concluding remarks

- The CSM approach has a broad range of applications and can usually provide certain particular solutions of hydrodynamic type equations.
- The new rank-2 and rank-3 periodic bounded solutions expressed in terms of the $\wp$-function represent bumps, anti-bumps and multiple-wave solutions.
- These solutions remain bounded even when the Riemann invariants admit the gradient catastrophe.
- We constructed the general rank-k solution for the isentropic fluid flow.
- Exact solutions may display qualitative behaviour which would otherwise be difficult to detect numerically or by approximations.
- A preliminary analysis shows that the conditional symmetry approach could be adapted for the analysis of elliptic quasilinear systems.