On the inverse problem of calculus of variations

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Abstract

We show that given any ordinary differential equation of even order n, it is always possible to determine at least one bona-fide Lagrangian if the $n-1$ derivative is absent (or eliminated) from the equation. The key is the Jacobi last multiplier as in the well-known case of a second-order equation. The known link between Jacobi last multiplier and Lie symmetries is exploited. Two equations from a Number Theory paper by Hall, one of second and one of fourth order, will be used to exemplify the method.

1 Introduction

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It is well-known that a Lagrangian always exists for any second-order ordinary differential equation [11]. The key is the Jacobi last multiplier [3], [4], [5], [11], which has many interesting properties, a list of which can be found in [9]. Here we use the Jacobi last multiplier in order to find Lagrangians for ordinary differential equations of even order $n > 2$. The known link between Jacobi last multiplier and Lie symmetries [6], [7] is exploited.

The paper is organized in the following way. In the next section we will present our method. In section 3 two equations from a Number Theory paper by Hall [2], one of second and one of fourth order, will be used to exemplify the method itself. The last section contains some final remarks.

2 The method

The well-known relationship between the Jacobi Last Multiplier, M, and the Lagrangian, $L = L(t, u, u')$, for any second-order equation

$$u'' = F(t, u, u') \tag{1}$$

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is [11]

$$M = \frac{\partial^2 L}{\partial u'^2} \tag{2}$$

where $M = M(t, u, u')$ satisfies the following equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} + M\frac{\partial F}{\partial u'} = 0. \tag{3}$$

Then equation (1) becomes the Euler-Lagrange equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial u'}\right) + \frac{\partial L}{\partial u} = 0. \tag{4}$$

The proof is given by taking the derivative of (4) by u' and showing that this yields (3). If one knows a Jacobi last multiplier, then L can be obtained by a double integration, i.e.:

$$L = \int\left(\int M\,\mathrm{d}u'\right)\mathrm{d}u' + f_1(t,u)u' + f_2(t,u), \tag{5}$$

where f_1 and f_2 are functions of t and u which have to satisfy a single partial differential equation related to (1) [10]. As it was shown in [10], f_1, f_2 are related to the gauge function $g = g(t, u, u')$. In fact, we may assume

$$\begin{aligned} f_1 &= \frac{\partial g}{\partial u} \\ f_2 &= \frac{\partial g}{\partial t} + f_3(t,u) \end{aligned} \tag{6}$$

where f_3 has to satisfy the mentioned partial differential equation and g is obviously arbitrary. The importance of the gauge function should be stressed. In order to apply Noether's theorem correctly, one should not assume $g \equiv const$, otherwise some first integrals may not be found (see [10] and the second-order equation in the next section).

We now consider a fourth-order equation, i.e.

$$u^{(iv)} = F(t, u, u', u'', u'''). \tag{7}$$

In this case the Jacobi last multiplier satisfies the following equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} + M\frac{\partial F}{\partial u'''} = 0. \tag{8}$$

It is easy to show that if a Lagrangian $L = L(t, u, u', u'')$ is taken such that

$$M = \frac{\partial^2 L}{\partial u''^2} \tag{9}$$

along with the constraint

$$\frac{\partial F}{\partial u'''} = 0, \tag{10}$$

then equation (7) becomes the Euler-Lagrange equation:

$$+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial u''}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial u'}\right) + \frac{\partial L}{\partial u} = 0. \tag{11}$$

The proof consists into taking the partial derivative of (11) by u''' and showing that this yields (8).

We underline that because of the assumption (10), then a Jacobi last multiplier M is easy to find from equation (8), namely

$$M = \mathrm{const.} \tag{12}$$

If one knows a Jacobi last multiplier, then L can be obtained by a double integration, i.e.:

$$L = \int \left(\int M \,\mathrm{d}u''\right) \mathrm{d}u'' + f_1(t, u, u')u'' + f_2(t, u, u'), \tag{13}$$

where f_1 and f_2 are functions of t, u, u' which have to satisfy some partial differential equations related to (7). We can relate f_1, f_2 to the gauge function $g = g(t, u, u', u'')$. In fact, we may assume

$$\begin{aligned} f_1 &= \frac{\partial g}{\partial u'} \\ f_2 &= \frac{\partial g}{\partial u}u' + \frac{\partial g}{\partial t} + f_3(t, u, u') \end{aligned} \tag{14}$$

where f_3 has to satisfy the mentioned partial differential equations and g is obviously arbitrary. Again we stress the importance of the gauge function. In order to apply Noether's theorem correctly, one should not assume $g \equiv const$, otherwise some first integrals may not be found (see the fourth-order equation in the next section).

We now consider a sixth-order equation, i.e.

$$u^{vi} = F(t, u, u', u'', u''', u^{iv}, u^{v}). \tag{15}$$

In this case the Jacobi last multiplier satisfies the following equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} + M\frac{\partial F}{\partial u^{v}} = 0. \tag{16}$$

It is easy to show that if a Lagrangian $L = L(t, u, u', u'', u''')$ is taken such that

$$M = \frac{\partial^2 L}{\partial u'''^2} \tag{17}$$

along with the constraint

$$\frac{\partial F}{\partial u^v} = 0, \tag{18}$$

then equation (15) becomes the Euler-Lagrange equation:

$$-\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(\frac{\partial L}{\partial u'''}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial u''}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial u'}\right) + \frac{\partial L}{\partial u} = 0. \tag{19}$$

The proof consists into taking the partial derivative of (19) by u^v and showing that this yields (16).

We underline that because of the assumption (18), then a Jacobi last multiplier M is easy to find from equation (16), namely

$$M = \mathrm{const.} \tag{20}$$

If one knows a Jacobi last multiplier, then L can be obtained by a double integration, i.e.:

$$L = \int \left(\int M \, \mathrm{d}u''' \right) \mathrm{d}u''' + f_1(t, u, u', u'')u''' + f_2(t, u, u', u''), \tag{21}$$

where f_1 and f_2 are functions of t, u, u', u'' which have to satisfy some partial differential equations related to (15). We can relate f_1, f_2 to the gauge function $g = g(t, u, u', u'', u''')$. In fact, we may assume

$$\begin{aligned} f_1 &= \frac{\partial g}{\partial u''} \\ f_2 &= \frac{\partial g}{\partial u'}u'' + \frac{\partial g}{\partial u}u' + \frac{\partial g}{\partial t} + f_3(t, u, u', u'') \end{aligned} \tag{22}$$

where f_3 has to satisfy the mentioned partial differential equations and g is obviously arbitrary. Again we stress the importance of the gauge function. In order to apply Noether's theorem correctly, one should not assume $g \equiv const$, otherwise some first integrals may not be found.

Finally, we consider any equation of order $2n > 2$, i.e.

$$u^{(2n)} = F(t, u, u', \ldots, u^{(2n-1)}). \tag{23}$$

In this case the Jacobi last multiplier satisfies the following equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} + M\frac{\partial F}{\partial u^{(n-1)}} = 0. \tag{24}$$

It is easy to show that if a Lagrangian $L = L(t, u, u', \ldots, u^{(n)})$ is taken such that

$$M = \frac{\partial^2 L}{\partial (u^{(n)})^2} \tag{25}$$

along with the constraint

$$\frac{\partial F}{\partial u^{(2n-1)}} = 0, \tag{26}$$

then equation (23) becomes the Euler-Lagrange equation:

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{\partial L}{\partial u^{(n)}}\right) + \ldots + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial u''}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial u'}\right) + \frac{\partial L}{\partial u} = 0. \tag{27}$$

The proof consists into taking the partial derivative of (27) by $u^{(2n-1)}$ and showing that this yields (24). In fact,

$$\frac{\partial}{\partial u^{(2n-1)}}\left((-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{\partial L}{\partial u^{(n)}}\right) + \ldots + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial u''}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial u'}\right) + \frac{\partial L}{\partial u}\right) =$$
$$(-1)^n n \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^2 L}{\partial (u^{(n)})^2}\right) + \frac{\partial^2 L}{\partial (u^{(n)})^2}\frac{\partial F}{\partial u^{(2n-1)}}. \tag{28}$$

We underline that because of the assumption (26), then a Jacobi last multiplier M is easy to find from equation (24), namely

$$M = \text{const.} \tag{29}$$

If one knows a Jacobi last multiplier, then L can be obtained by a double integration, i.e.:

$$L = \int\left(\int M\,\mathrm{d}u^{(n)}\right)\mathrm{d}u^{(n)} + f_1(t, u, u', \ldots, u^{(n-1)})u^{(n)} + f_2(t, u, u', \ldots, u^{(n-1)}), \tag{30}$$

where f_1 and f_2 are functions of $t, u, u', \ldots, u^{(n-1)}$ which have to satisfy some partial differential equations related to (23). We can relate f_1, f_2 to the gauge function $g = g(t, u, u', \ldots, u^{(n-1)})$. In fact, we may assume

$$\begin{aligned} f_1 &= \frac{\partial g}{\partial u^{(n-1)}} \\ f_2 &= \frac{\partial g}{\partial u^{(n-2)}}u^{(n-1)} + \frac{\partial g}{\partial u^{(n-3)}}u^{(n-2)} + \ldots + \frac{\partial g}{\partial t} + f_3(t, u, u', \ldots, u^{(n-1)}) \end{aligned} \tag{31}$$

where f_3 has to satisfy the mentioned partial differential equations and g is obviously arbitrary. Again we stress the importance of the gauge function. In order to apply Noether's theorem correctly, one should not assume $g \equiv const$, otherwise some first integrals may not be found.

3 Two examples from Number Theory

3.1 A second-order equation

In [2] the following functional was introduced:

$$\int_0^\pi y'^4 + 6\nu y^2 y'^2 d\,x, \tag{32}$$

where $y = y(x) \in C^2[0,\pi]$, $y(0) = y(\pi) = 0$ and $\nu \geq 0$. The corresponding Euler-Lagrange equation is

$$y'^2 y'' + \nu y^2 y'' + \nu y y'^2 = 0\,. \tag{33}$$

If we apply Lie group analysis to this equation, we find that it admits a two-dimensional abelian transitive Lie symmetry algebra (Type I) generated by the following two operators:

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = y\partial_y\,. \tag{34}$$

Then we can integrate equation (33). First, we introduce a basis of differential invariants of Γ_1, i.e.:

$$u = \frac{y'}{y}, \quad v = x\,. \tag{35}$$

Then equation (33) reduces to the following first-order equation:

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{-\nu u v}{\nu v^2 + u^2}, \tag{36}$$

which admits the operator Γ_2 in the space of variables u, v, i.e.

$$\Gamma_2 = v\partial_v + u\partial_u\,. \tag{37}$$

Then its general solution is implicitly given by:

$$\sqrt{u}\left(2\nu v^2 + u^2\right)^{1/4} = const \tag{38}$$

and in the original variables[1]:

$$\sqrt{y'}\left(2\nu y^2 + y'^2\right)^{1/4} = const \tag{39}$$

viz

$$y' = \frac{\sqrt{\left(\nu y^2 a_1 + \sqrt{\nu^2 y^4 a_1^2 + 1}\right) y^2 a_1}}{y a_1 \left(\nu y^2 a_1 + \sqrt{\nu^2 y^4 a_1^2 + 1}\right)} \tag{40}$$

with a_1 an arbitrary constant Finally the general solution of (33) is given implicitly by:

$$\int \frac{y a_1 \left(\nu y^2 a_1 + \sqrt{\nu^2 y^4 a_1^2 + 1}\right)}{\sqrt{\left(\nu y^2 a_1 + \sqrt{\nu^2 y^4 a_1^2 + 1}\right) y^2 a_1}}\, d\,y = x + a_2 \tag{41}$$

[1] The same first integral can be obtained by using Noether's theorem (see below)

We note that if y is positive and $a_1 = 1$ the integral on the left-hand side could be integrated in terms of an hypergeometric function $\mathcal{H}$, namely:

$$\frac{1}{2}\sqrt{2\nu}y^2\mathcal{H}\left(\left[-\frac{1}{2},\frac{1}{4},-\frac{1}{4}\right],\left[\frac{1}{2},\frac{1}{2}\right],-\frac{1}{y^4\nu^2}\right) \tag{42}$$

Let us try to find a Lagrangian for equation (33) by using the Jacobi last multiplier, namely through (2). The two Lie point symmetries (34) yield a Jacobi last multiplier. In fact the following matrix [6], [7]:

$$\begin{pmatrix} 1 & y' & -\frac{\nu y y'^2}{\nu} y^2 + y'^2 \\ 1 & 0 & 0 \\ 0 & y & y' \end{pmatrix} \tag{43}$$

has determinant different from zero and its inverse is a Jacobi last multiplier, i.e.

$$M_1 = -\frac{2\nu y^2 + y'^2}{y'^2(\nu y^2 + y'^2)}. \tag{44}$$

The corresponding Lagrangian is

$$L_1 = \frac{1}{4\nu y}\left(-\sqrt{2\nu}y' \arctan\left(\frac{y'}{\sqrt{2\nu}y}\right) + \log(2\nu y^2 + y'^2)\nu y + 2\log(y')\nu y\right) + f_1(x,y)y' + f_2(x,y), \tag{45}$$

where f_1, f_2 are solutions of

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} = 0. \tag{46}$$

If we impose the link between f_1, f_2 with the gauge function $g(x,y)$, namely (6), then $f_3(x,y)$ becomes just $f_3(x)$, an arbitrary function of the independent variable x. The Lagrangian (45) may appear ugly. Nevertheless the corresponding variational problem admits two Noether's symmetries, namely both Lie symmetries given in (34) are Noether's symmetries. Consequently the following two first integrals of equation (33) can be found by applying Noether theorem[2]:

$$\Gamma_1 \quad \Rightarrow \quad I_1 = (2\nu y^2 + y'^2)y'^2, \qquad \left[g = e^x\left(\int \frac{f_3(x)}{e^x}\,\mathrm{d}x + a_2\right)\right] \tag{47}$$

$$\Gamma_2 \quad \Rightarrow \quad I_2 = \frac{1}{4\nu y'}\left(-\sqrt{(}2\nu)y' \arctan\left(\frac{y'}{\sqrt{2\nu}y}\right) + 2\nu y - 4\nu x y'\right), \quad [g = s(x)y + x]. \tag{48}$$

with $s(x)$ an arbitrary function of x. We note that the first integral I_1 in (47) was already derived in (39).

[2]Also the corresponding gauge function g is given. It is important to remark that in the case of the first integral (48) the gauge function g cannot be constant, while it can be constant in the case of the first integral (47). Naturally, we have left out any inessential additive constants.

At this point one would like to know if it is possible to obtain the original Lagrangian given in (32), i.e.

$$L_H = y'^4 + 6\nu y^2 y'^2 \tag{49}$$

A property of the Jacobi last multiplier is that if one knows a Jacobi last multiplier and a first integral then their product gives another multiplier [9]. If we take the product of the first integral I_1 (47) and the multiplier M_1 (44), then we obtain another Jacobi last multiplier of equation (33), i.e.:

$$M_2 = -y'^2 - \nu y^2 \tag{50}$$

which can be integrated twice with respect to y' in order to yield the following Lagrangian[3]:

$$L_2 = -\frac{1}{12}(y'^4 + 6\nu y^2 y'^2) + f_1 y' + f_2, \tag{51}$$

where f_1, f_2 are solutions of (46). It is interesting to emphasize that this Lagrangian (namely Hall's Lagrangian) is such that the Lie operator Γ_2 in (34) does not generate a Noether's symmetry for the corresponding variational problem. In fact only Γ_1 is the generator of a Noether's symmetry for Hall's Lagrangian.

If instead of (32) we consider the functional [2]

$$\int_0^\pi y'^4 + 6\nu y^2 y'^2 - 3\lambda(\nu) y^4 \, d\,x. \tag{52}$$

and apply Lie group analysis to its corresponding Euler-Lagrange equation, viz:

$$y'^2 y'' + \nu y^2 y'' + \nu y y'^2 + \lambda y^3 = 0\,, \tag{53}$$

then we obtain the same Lie symmetry algebra generated by (34) which means that (53) can be integrated by quadrature. In fact if we introduce the same variables as in (35) then equation (53) reduces to the following first-order equation:

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{-v\left(\lambda v^2 + \nu u^2\right)}{u\left(\nu v^2 + u^2\right)}, \tag{54}$$

which can be integrated to give

$$\lambda v^4 + 2\nu u^2 v^2 + u^4 = const \tag{55}$$

and in the original variables

$$\lambda y^4 + 2\nu y'^2 y^2 + y'^4 = const \tag{56}$$

[3]Note the inessential multiplicative constant.

a first integral of equation (53).

Also Lie group analysis implies that if $\lambda = \nu^2$ then equation (53) admits an eight-dimensional Lie symmetry algebra generated by the following operators:

$$\begin{aligned}
\Gamma_1 &= -y\left(-\cos(\sqrt{\nu}x)\partial_x + y\sqrt{\nu}\sin(\sqrt{\nu}x)\partial_y\right)\\
\Gamma_2 &= -y\left(\sin(\sqrt{\nu}x)\partial_x + y\sqrt{\nu}\cos(\sqrt{\nu}x)\partial_y\right)\\
\Gamma_3 &= \cos(2\sqrt{\nu}x)\partial_x - y\sqrt{\nu}\sin(2\sqrt{\nu}x)\partial_y\\
\Gamma_4 &= -\sin(2\sqrt{\nu}x)\partial_x - y\sqrt{\nu}\cos(2\sqrt{\nu}x)\partial_y\\
\Gamma_5 &= \partial_x\\
\Gamma_6 &= y\partial_y\\
\Gamma_7 &= \cos(\sqrt{\nu}x)\partial_y\\
\Gamma_8 &= -\sin(\sqrt{\nu}x)\partial_y.
\end{aligned} \tag{57}$$

which means that equation (53) is linearizable or indeed linear. Indeed in this case equation (53) is just

$$y'' = -\nu y. \tag{58}$$

In order to find Lagrangians and first integrals for equation (52) we have to repeat mutatis mutandis the same analysis given above. We may underline that in the case of $\lambda = \nu^2$, namely equation (58), we can generate 14 different Lagrangians as it was shown in [10].

3.2 A fourth-order equation

Another functional in [2] is the following:

$$\int_0^\pi y'^4 + \mu y^2 y''^2 \, d\, x\,. \tag{59}$$

The corresponding Euler-Lagrange equation is

$$4\mu y y' y''' + \mu y^2 y^{iv} + 2\mu y'^2 y'' + 3\mu y y''^2 - 6y'^2 y'' = 0\,. \tag{60}$$

If we apply Lie group analysis to this equation, we find that it admits a three-dimensional Lie symmetry algebra generated by the following three operators:

$$X_1 = \partial_x, \quad X_2 = y\partial_y, \quad X_3 = x\partial_x\,, \tag{61}$$

which means that we can reduce equation (60) to a first-order equation, i.e.:

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{x}} = \frac{-7\mu\tilde{u}\tilde{x} - \mu\tilde{u} - 6\mu\tilde{x}^3 - 4\mu\tilde{x}^2 + 6\tilde{x}}{\mu\tilde{u}} \tag{62}$$

with

$$\tilde{u} = \frac{y'''y^2}{y'^3} - 2\frac{y''^2y^2}{y'^4} + \frac{y''y}{y'^2}, \quad \tilde{x} = \frac{y''y}{y'^2}\,. \tag{63}$$

If $\mu = 3$ then equation (60) admits an eight-dimensional Lie symmetry algebra $\mathcal{L}$ generated by the following eight operators:

$$\Lambda_1 = x^2\partial_x + \frac{3}{2}xy\partial_y, \qquad \Lambda_2 = x\partial_x, \qquad \Lambda_3 = \partial_x, \qquad \Lambda_4 = y\partial_y, \qquad \Lambda_5 = \frac{x^3}{y}\,\partial_y,$$
$$\Lambda_6 = \frac{x^2}{y}\,\partial_y, \qquad \Lambda_7 = \frac{x}{y}\,\partial_y, \qquad \Lambda_8 = \frac{1}{y}\,\partial_y \tag{64}$$

This means that equation (60), i.e.:

$$4y'y''' + yy^{iv} + 3y''^2 = 0 \tag{65}$$

is linearizable by means of a point transformation [7]. In order to find the linearizable transformation we have to find an abelian intransitive two-dimensional subalgebra of $\mathcal{L}$ and, following Lie's classification of two-dimensional algebras in the real plane [7], we have to transform it into the canonical form

$$\partial_u, \qquad t\partial_u \tag{66}$$

with u and t the new dependent and independent variables, respectively. We found that one such subalgebra is that generated by Λ_7 and Λ_8. Then we have to solve the following four linear partial differential equations of first order:

$$\Lambda_7(t) = 0, \quad \Lambda_8(t) = 0, \quad \Lambda_7(u) = t, \quad \Lambda_8(u) = 1. \tag{67}$$

It is readily shown that the linearizable transformation is

$$t = x, \qquad u = y^2, \tag{68}$$

and equation (65) becomes

$$u^{iv} = 0. \tag{69}$$

Finally, the general solution of (65) is

$$y = \sqrt{a_1 + a_2x + a_3x^2 + a_4x^3} \tag{70}$$

with $a_i (i = 1, 4)$ arbitrary constants.

We note that if we apply the transformation (68) to equation (60) in the case of any μ then the following equation is obtained:

$$u^{iv} = -\frac{(\mu - 3)(2uu'' - u'^2)u'^2}{4\mu u^3} \tag{71}$$

which does not contain u''', namely the third derivative of u by x. Therefore a constant, say 1, is a Jacobi last multiplier of equation (71), and we can obtain a Lagrangian from (21), i.e.:

$$L_1 = \frac{1}{2}u''^2 + f_1(x,u,u')u'' + f_2(x,u,u') \tag{72}$$

where f_1 and f_2 satisfy (22) and f_3 has to have the following expression:

$$f_3(x,u,u') = \frac{(-\mu+3)u'^4}{24\mu u^2} + h_1(x,u)u' + h_2(x,u) \tag{73}$$

with h_1, h_2 arbitrary functions of x, u.

We remark that a Jacobi last multiplier of equation (60) can be derived from equation (8), i.e.:

$$\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}x} - 4\frac{y'}{y} = 0 \quad \Longrightarrow \quad M = y^4, \tag{74}$$

although this Jacobi last multiplier is useless in order to find a Lagrangian of equation (60). Instead, if we apply transformation (68) to (72) in order to go back to the original function $y(x)$, then adding also the particular assumptions $f_1 = -u'^2/(2u)$, and $f_2 = u'^4(\mu+1)/(16u^2)$ yield the following Lagrangian for equation (60) :

$$Lm_1 = (\mu-1)y'^4 + 2y^2y''^2 \tag{75}$$

which, apart from an inessential multiplicative constant, is Hall's Lagrangian in (59).
Let us apply Noether's theorem. If we consider the Lagrangian in (72), we find that the following two first integrals of equation (71) can be obtained:

$$\begin{aligned}
\frac{3}{2}X_3 + X_1 \quad &\Rightarrow \quad I_1 = \frac{1}{8\mu u^2}\left(12\mu u^3u''' - 4\mu u^2u'u'' - 8\mu u^2u'u'''x + 4\mu u^2u''^2x\right.\\
&\qquad\qquad \left. +2\mu uu'^3 - \mu u'^4x - 6uu'^3 + 3u'^4x\right),\\
X_2 \quad &\Rightarrow \quad I_2 = \frac{1}{8\mu u^2}(-8\mu u^2u'u''' + 4\mu u^2u''^2 - \mu u'^4 + 3u'^4).
\end{aligned} \tag{76}$$

A similar result is obtained if one uses Hall's Lagrangian and equation (60). Moreover, if we apply Noether's theorem to the linearizable equation (65) with Hall's Lagrangian, i.e.:

$$L_H = y'^4 + 3y^2y''^2, \tag{77}$$

we obtain the following seven first integrals[4]:

$$\Lambda_1 \quad \Rightarrow \quad Im_1 = -3y^3y'' + 3y^3y'''x + y^2y'^2 + 7y^2y'y''x$$

[4]Naturally, they are not all independent from each other.

$$-2y^2y'y'''x^2 + y^2y''^2x^2 - 2yy'^3x - 4yy'^2y''x^2 + y'^4x^2$$

$$\begin{array}{lcl}
\frac{3}{4}\Lambda_4 + \Lambda_2 & \Rightarrow & Im_2 = 9y^3y''' + 21y^2y'y'' - 12y^2y'y'''x + 6y^2y''^2x - 6yy'^3 - 24yy'^2y''x + 6y'^4x \\
\Lambda_3 & \Rightarrow & Im_3 = -2y^2y'y''' + y^2y''^2 - 4yy'^2y'' + y'^4 \\
\Lambda_5 & \Rightarrow & Im_5 = -3y^2 + 6yy'x - 3yy''x^2 + yy'''x^3 - 3y'^2x^2 + 3y'y''x^3 \\
\Lambda_6 & \Rightarrow & Im_6 = 2yy' - 2yy''x + yy'''x^2 - 2y'^2x + 3y'y''x^2 \\
\Lambda_7 & \Rightarrow & Im_7 = -yy'' + yy'''x - y'^2 + 3y'y''x \\
\Lambda_8 & \Rightarrow & Im_8 = yy''' + 3y'y''
\end{array} \tag{78}$$

Although we do not write down the corresponding expressions of the gauge function, we underline that it cannot always be set equal to a constant otherwise none of Im_1, Im_5, Im_6, Im_7, and Im_8 could be obtained.

All seven first integrals (and even more) may be obtained without Noether's Theorem. In fact we just need to find the Jacobi last multipliers of equation (65) that are obtained by inverting the nonzero determinants of the possible 70 matrices made out of the eight Lie symmetries (64). Then the ratio of any two multipliers is a first integral of equation (65). For example:

$$C_{1234} = \begin{pmatrix} 1 & y' & y'' & y''' & -\dfrac{4y'y''' + 3y''^2}{y} \\ x^2 & \frac{3}{2}yx & \frac{3}{2}y - \frac{1}{2}xy' & y' - \frac{5}{2}xy'' & -\frac{3}{2}y'' - \frac{9}{2}xy''' \\ x & 0 & -y' & -2y'' & -3y''' \\ 1 & 0 & 0 & 0 & 0 \\ 0 & y & y' & y'' & y''' \end{pmatrix} \tag{79}$$

is the matrix obtained by considering the symmetries generated by operators $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ in (64); its determinant is

$$\Delta_{1234} = 9y''yy'y''' - 2y'^3y''' - \frac{3}{2}y'^2y''^2 + 6y''^3y + \frac{9}{2}y'''^2y^2, \tag{80}$$

and the corresponding Jacobi last multiplier is:

$$M_{1234} = \frac{1}{\Delta_{1234}} = \frac{1}{18y''yy'y''' - 4y'^3y''' - 3y'^2y''^2 + 12y''^3y + 9y'''^2y^2}. \tag{81}$$

Similarly, the matrix C_{5678} yields the determinant $\Delta_{5678} = 12/y^4$, i.e. the Jacobi last multiplier[5]:

$$M_{5678} = y^4 \tag{82}$$

[5]The multiplicative constant is inessential.

which we have already found in (74) as an obvious solution of (8). Note that

$$\frac{M_{5678}}{M_{1234}} = y^4(18y''yy'y''' - 4y'^3y''' - 3y'^2y''^2 + 12y''^3y + 9y'''^2y^2) \tag{83}$$

is "another" first integral of equation (65).

4 Final remarks

DRAFT

The following remarks should be enlightened and kept in mind:

- The most efficient Lagrangian, namely that which allows the most number of Noether's symmetries, may not be the Lagrangian with the simplest form.
- Lie symmetries are the key tool for finding Jacobi last multipliers and therefore Lagrangians.

In [1] the necessary and sufficient conditions under which a fourth-order equation (7) admits a unique Lagrangian were determined, namely:

$$\frac{\partial^3 F}{\partial (u''')^3} = 0 \tag{84}$$

$$\frac{\partial F}{\partial u'} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial F}{\partial u'''}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial u''}\right) - \frac{3}{4}\frac{\partial F}{\partial u'''}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial u'''}\right) + \frac{1}{2}\frac{\partial F}{\partial u''}\frac{\partial F}{\partial u'''} + \frac{1}{8}\left(\frac{\partial F}{\partial u'''}\right)^3 = 0. \tag{85}$$

Not surprisingly both equation (60) and (71) satisfy those two conditions.

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