Lagrangians for biological models: the method of Jacobi Last Multiplier

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Abstract

We derive Lagrangians of classic models in biology, such the Lotka-Volterra system.

1 Introduction

"Among the mathematical results obtained by studying the inverse problem of mechanics, it is the explicit algorithms for constructing Lagrangians that offer the model builder the most practical benefit." [14].

It is well-known that a Lagrangian always exists for any second-order ordinary differential equation [18].

[11] and the references within.

"What are the criteria that a system of ordinary differential equations must satisfy to assure the existence of a Lagrangian?" [14].

"Does there exist an algorithm that enables one to construct the Lagrangian from the dynamical equations?" [14].

p.756 equation (27) [14] g is actually the JLM of system (21) for n=2

"These results clearly show that as the interaction between the populations becomes more complex the corresponding Lagrangian becomes more complicated and difficult to find" [16].

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2 The method by Jacobi

The method of the Jacobi last multiplier ([2], [3], [4], [5]) provides a means to determine all the solutions of the partial differential equation

$$\mathcal{A}f = \sum_{i=1}^{n} a_i(x_1, \dots, x_n) \frac{\partial f}{\partial x_i} = 0 \tag{1}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \tag{2}$$

In fact, if one knows the Jacobi last multiplier and all but one of the solutions, then the last solution can be obtained by a quadrature. The Jacobi last multiplier M is given by

$$\frac{\partial(f, \omega_1, \omega_2, \dots, \omega_{n-1})}{\partial(x_1, x_2, \dots, x_n)} = M\mathcal{A}f, \tag{3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \dots, \omega_{n-1})}{\partial(x_1, x_2, \dots, x_n)} = \det \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{4}$$

and $\omega_1, \dots, \omega_{n-1}$ are $n-1$ solutions of (1) or, equivalently, first integrals of (2) independent of each other. This means that M is a function of the variables $(x_1, \dots, x_n)$ and depends on the chosen $n-1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi last multiplier are:

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \dots, \eta_{n-1}$ of equation (1), then the corresponding last multiplier N is linked to M by the relationship:

$$N = M \frac{\partial(\eta_1, \dots, \eta_{n-1})}{\partial(\omega_1, \dots, \omega_{n-1})}.$$

(b) Given a non-singular transformation of variables

$$\tau : \quad (x_1, x_2, \dots, x_n) \longrightarrow (x'_1, x'_2, \dots, x'_n),$$

then the last multiplier M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x'_1, x'_2, \ldots, x'_n)},$$

where M obviously comes from the $n-1$ solutions of $\mathcal{A}F = 0$ which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1}.

(c) One can prove that each multiplier M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0; \tag{5}$$

viceversa every solution M of this equation is a Jacobi last multiplier.

(d) If one knows two Jacobi last multipliers M_1 and M_2 of equation (1), then their ratio is a solution ω of (1), or, equivalently, a first integral of (2). Naturally the ratio may be quite trivial, namely a constant. Viceversa the product of a multiplier M_1 times any solution ω yields another last multiplier $M_2 = M_1\omega$.

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, then any other Jacobi Last Multiplier is a first integral.

Another property of the Jacobi Last Multiplier is its (almost forgotten) relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation

$$\ddot{x} = F(t, x, \dot{x}) \tag{6}$$

is [5], [18]

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} \tag{7}$$

where $M = M(t, x, \dot{x})$ satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \frac{\partial F}{\partial \dot{x}} = 0. \tag{8}$$

Then equation (6) becomes the Euler-Lagrangian equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0. \tag{9}$$

The proof is given by taking the derivative of (9) by $\dot{x}$ and showing that this yields (8). If one knows a Jacobi last multiplier, then L can be easily obtained by a double integration, i.e.:

$$L = \int \left(\int M \,\mathrm{d}\dot{x} \right) \mathrm{d}\dot{x} + f_1(t, x)\dot{x} + f_2(t, x), \tag{10}$$

where f_1 and f_2 are functions of t and x which have to satisfy a single partial differential equation related to (6) [12]. As it was shown in [12], f_1, f_2 are related to the gauge function $F = F(t, x)$. In fact, we may assume

$$\begin{aligned} f_1 &= \frac{\partial F}{\partial x} \\ f_2 &= \frac{\partial F}{\partial t} + f_3(t, x) \end{aligned} \tag{11}$$

where f_3 has to satisfy the mentioned partial differential equation and F is obviously arbitrary.

3 Some biological examples from [16]: Non Linear Lagrangians for second-order equations

3.1 Volterra-Lotka

The Volterra-Lotka model considered by [16] is the following:

$$\begin{aligned} \dot{w}_1 &= w_1(a + bw_2) \\ \dot{w}_2 &= w_2(A + Bw_1). \end{aligned} \tag{12}$$

We note that there is an obvious autonomous first integral, namely

$$I_1 = Bw_1 - bw_2 + A - a\log(w_2) + A\log(-Bw_1) \tag{13}$$

which can be obtained by a simple quadrature from the equation

$$\frac{\mathrm{d}w_1}{\mathrm{d}w_2} = \frac{w_1(a + bw_2)}{w_2(A + Bw_1)}. \tag{14}$$

In order to simplify system (12) we follow [16] and introduce the change of variables

$$w_1 = \exp(r_1), \qquad w_2 = \exp(r_2) \tag{15}$$

and then system (12) becomes

$$\begin{aligned} \dot{r}_1 &= b\exp(r_2) + a \\ \dot{r}_2 &= B\exp(r_1) + A. \end{aligned} \tag{16}$$

We can transform this system into an equivalent second-order ordinary differential equation by eliminating, say, r_1. In fact from the second equation in (16) one gets

$$r_1 = \log\left(\frac{\dot{r}_2 - A}{B}\right), \tag{17}$$

and the equivalent second-order equation in r_2 is the following

$$\ddot{r}_2 = -\Big(b\exp(r_2) + a\Big)(A - \dot{r}_2). \tag{18}$$

A Jacobi Last Multiplier for this equation has to satisfy equation (8), i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + b\exp(r_2) + a = 0 \tag{19}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \dot{r}_1 = 0, \tag{20}$$

by taking into account the first equation in (16), and consequently we get the following Jacobi Last Multiplier for equation (18):

$$M_1 = \exp(-r_1) = \frac{B}{\dot{r}_2 - A}, \tag{21}$$

the last equality being true thanks to (17). Then a Lagrangian can be easily obtained by a double integration as in (10), i.e.

$$L_1 = B\Big((\dot{r}_2 - A)\log(A - \dot{r}_2) - \dot{r}_2 + b\exp(r_2) + ar_2\Big) + \dot{F}(t, r_2). \tag{22}$$

The same Lagrangian (minus the gauge function F) was obtained in [16] by using an ad hoc method. In order to show the power of the Jacobi's method we derive at least another Lagrangian for equation (18).

We note that (18) is autonomous and therefore invariant under time translation. It is easy to show that the Lagrangian L_1 in (22) yields a time-invariant first integral, namely (13), through Noether's theorem [10], which yields the following first integral if L is invariant under time translation:

$$-L + \dot{x}\frac{\partial L}{\partial \dot{x}} + F(t, x) = const \tag{23}$$

As a consequence of the property (d) of the Jacobi last multiplier, the product of a Jacobi last multiplier M_1 as in (21) and a first integral I_1 as in (13) of equation (18) yields another Jacobi last multiplier, i.e.

$$M_2 = M_1 I_1 = \frac{B^2}{A - \dot{r}_2}\Big(ar_2 - \dot{r}_2 - A\log(A - \dot{r}_2) + b\exp(r_2)\Big) \tag{24}$$

and therefore we can obtain a second Lagrangian of equation (18), i.e.

$$\begin{aligned} L_2 \;=\; & -\frac{B^2}{2}\Big((A\log(A - \dot{r}_2) - 2ar_2)\,(A - \dot{r}_2)\log(A - \dot{r}_2) \\ & -(2ar_2 + \dot{r}_2)\dot{r}_2 - 2b\exp(r_2)\,((A - \dot{r}_2)\log(A - \dot{r}_2) + \dot{r}_2) \\ & + b^2\exp(2r_2) + 2abr_2\exp(r_2) + a^2 r_2^2\Big) + \dot{F}(t, r_2) \end{aligned} \tag{25}$$

This Lagrangian yields another time invariant first integral which is just the square of I_1 in (13).
We can keep using property (d) to derive more and more Jacobi last multipliers and therefore Lagrangians of equation (18). In fact other Jacobi last multipliers can be obtained by simply taking any function of the first integral I_1 in (13) and multiplying it for either M_1 in (21) or M_2 in (24), and so on ad libitum.

We would like to point out that a Jacobi Last Multiplier for system (16) is a constant, and therefore by using property (b) we can easily derive a Jacobi Last Multiplier for the Volterra-Lotka system (12). In fact we have to calculate the Jacobian of the transformation (15) between (w_1, w_2) and (r_1, r_2) and this yields a Jacobi Last Multiplier of system (12), i.e.

$$M_{[w]} = M_{[r]} \frac{\partial(r_1, r_2)}{\partial(w_1, w_2)} = \begin{vmatrix} \dfrac{1}{w_1} & 0 \\ 0 & \dfrac{1}{w_2} \end{vmatrix} = \frac{1}{w_1 w_2}. \tag{26}$$

Then the system (12) is completely integrable as it was shown by Jacobi himself [1]: a system of two first-order ordinary differential equations is completely solved when one knows a first integral and a Jacobi Last Multiplier [5].

3.2 Gompertz

The Gompertz's model considered by [16] is the following:

$$\begin{aligned} \dot{w}_1 &= w_1 \left(A \log\left(\frac{w_1}{m_1}\right) + B w_2 \right) \\ \dot{w}_2 &= w_2 \left(a \log\left(\frac{w_2}{m_2}\right) + b w_1 \right). \end{aligned} \tag{27}$$

In order to simplify system (27) we follow [16] and introduce the change of variables

$$w_1 = m_1 \exp(r_1), \qquad w_2 = m_2 \exp(r_2) \tag{28}$$

and then system (27) becomes

$$\begin{aligned} \dot{r}_1 &= m_2 B \exp(r_2) + A r_1 \\ \dot{r}_2 &= m_1 b \exp(r_1) + a r_2. \end{aligned} \tag{29}$$

It is easy to derive a Jacobi Last Multiplier for this system from (5), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \log\left(M_{[r]}\right) = -(a + A) \Longrightarrow M_{[r]} = \exp[(-a + A)t] \tag{30}$$

We can transform system (29) into an equivalent second-order ordinary differential equation by eliminating, say, r_2. In fact from the second equation in (29) one gets

$$r_2 = \log\left(\frac{\dot{r}_1 - Ar_1}{Bm_2}\right), \tag{31}$$

and the equivalent second-order equation in r_2 is the following

$$\ddot{r}_1 = \left(bm_1 \exp(r_1) + a\log\left(\frac{\dot{r}_1 - Ar_1}{Bm_2}\right)\right)(\dot{r}_1 - Ar_1) + A\dot{r}_1. \tag{32}$$

Using property (b) a Jacobi Last Multiplier for this equation can be obtained. In fact we have to calculate the Jacobian of the transformation between (r_1, r_2) and $(r_1, \dot{r}_1)$, namely (31) and this yields a Jacobi Last Multiplier of equation (32), i.e.[1]

$$M_1 = M_{[r]}\frac{\partial(r_1, r_2)}{\partial(r_1, \dot{r}_1)} = \exp[-(a+A)t]\frac{1}{\dot{r}_1 - Ar_1}. \tag{33}$$

Then a Lagrangian can be easily obtained by a double integration as in (10), i.e.

$$L_1 = \exp[-(a+A)t]\Big((\dot{r}_1 - Ar_1)\log(\dot{r}_1 - Ar_1) + m_1 b\exp(r_1) - ar_1\log(Bm_2) - ar_1\Big) + \dot{F}(t, r_1). \tag{34}$$

The same Lagrangian (minus the gauge function F) was obtained in [16] by using an ad hoc method.
Finally property (b) yields a Jacobi Last Multiplier for the Gompertz's system (27). The product of $M_{[r]}$ in (30) with the Jacobian of the transformation (28) between (w_1, w_2) and (r_1, r_2) yields the following Jacobi Last Multiplier of system (27), i.e.

$$M_{[w]} = M_{[r]}\frac{\partial(r_1, r_2)}{\partial(w_1, w_2)} = \exp[-(a+A)t]\begin{vmatrix} \frac{1}{w_1} & 0 \\ 0 & \frac{1}{w_2} \end{vmatrix} = \exp[-(a+A)t]\frac{1}{w_1 w_2}. \tag{35}$$

3.3 Verhulst

The Verhulst's model considered by [16] is the following:

$$\begin{aligned} \dot{w}_1 &= w_1(A + Bw_1 + f_1 w_2) \\ \dot{w}_2 &= w_2(a + bw_2 + f_2 w_1). \end{aligned} \tag{36}$$

In order to derive a Jacobi Last Multiplier for this system from (5), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\left(M_{[w]}\right) + (2B + f_2)w_1 + (2b + f_1)w_2 + a + A = 0 \tag{37}$$

[1]Of course, we do not consider any multiplicative constants because they are inessential.

we assume that $M_{[w]}$ has the following form:

$$M_{[w]} = w_1^{b_1} w_2^{b_2} exp(b_3 t), \tag{38}$$

where $b_i, (i = 1, 2, 3)$ are constants to be determined. Replacing this $M_{[w]}$ into (37) yields

$$\begin{aligned} b_1 &= \frac{-2Bb + bf_2 + f_1 f_2}{Bb - f_1 f_2} & (39) \\ b_2 &= \frac{-2Bb + Bf_1 + f_1 f_2}{Bb - f_1 f_2} & (40) \\ b_3 &= \frac{ABb - Abf_2 + aBb - aBf_1}{Bb - f_1 f_2}, & (41) \end{aligned}$$

if $Bb - f_1 f_2 \neq 0$, and therefore if no condition is imposed on the parameters in Verhulst's model.

We follow [16] and introduce the change of variables

$$w_1 = \exp(r_1), \qquad w_2 = \exp(r_2) \tag{42}$$

and then system (36) becomes

$$\begin{aligned} \dot{r}_1 &= A + B\exp(r_1) + F\exp(r_2) \\ \dot{r}_2 &= a + b\exp(r_2) + f\exp(r_1). \end{aligned} \tag{43}$$

We can transform this system into an equivalent second-order ordinary differential equation by eliminating, say, r_2. In fact from the second equation in (46) one gets

$$r_1 =, \tag{44}$$

and the equivalent second-order equation in r_1 is the following

$$\ddot{r}_1 = . \tag{45}$$

3.4 Host-Parasite

A simple mathematical model which describes the interaction between a host and its parasite and which takes into account the non-linear effects of the host population size on the growth rate of the parasite population is given by the equations (Leslie & Gower, 1960)

$$\begin{aligned} \dot{H} &= (a - bP)H \\ \dot{P} &= \left(A - B\frac{P}{H}\right)P. \end{aligned} \tag{46}$$

4 Final remarks

AS stated by Paine [14], “This gives one hope of finding an integral or constant of motion for the dynamical system of interest without the hardship of solving the system of equations.”

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References

[1] Jacobi C G J, Sur un noveau principe de la mécanique analytique, *C. R. Acad. Sci. Paris* **15** (1842), 202–205.

[2] Jacobi C G J, Sul principio dell’ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica, *Giornale Arcadico di Scienze, Lettere ed Arti* **99** (1844), 129–146.

[3] Jacobi C G J, Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi, *J. Reine Angew. Math.* **27** (1844), 199–268.

[4] Jacobi C G J, Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi, *J. Reine Angew. Math.* **29** (1845), 213–279 and 333–376.

[5] Jacobi C G J, Vorlesungen über Dynamik. Nebst fünf hinterlassenen Abhandlungen desselben herausgegeben von A. Clebsch, Druck und Verlag von Georg Reimer, Berlin, 1886.

[6] Leslie, P. H. and Gower, J. C. (1960). Biometrika 47, 219.

[7] Lie S, Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen. Leipzig: Teubner; 1912.

[8] Lie S, Veralgemeinerung und neue Verwerthung der Jacobischen Multiplicator-Theorie. Fordhandlinger i Videnokabs-Selshabet i Christiania, 1874;255-274.

[9] Lotka, A. J. 1956. Elements of Mathematical Biology. New York: Dover.

[10] Noether E. Invariante Variationsprobleme. Nachr d König Gesellsch d Wiss zu Göttingen Math-phys Klasse, 1918;235-257.

[11] M.C. Nucci, "Jacobi last multiplier and Lie symmetries: a novel application of an old relationship" *J. Nonlinear Math. Phys. 12* (2005) pp. 284–304.

[12] M.C. Nucci and P.G.L. Leach, "Lagrangians galore", *J. Math. Phys. 48* (2007) 123510 (16 pages).

[13] M.C. Nucci and K.M. Tamizhmani, Using an old method of Jacobi to derive Lagrangians: a nonlinear dynamical system with variable coefficients, preprint

[14] G. H. Paine, The development of Lagrangians for biological models, Bulletin of Mathematical Biology 44, 749-760, 1982.

[15] Stein, R. B., K. V. Leung, M. N. Ogŭztöreli and D. W. Williams. 1974. "Properties of Small Neural Networks." Kybernetik 14, 223-230.

[16] S. L. Trubatch and A. Franco. Canonical Procedures for Population Dynamics J. Theor. Biol. (1974) 48, 299-324

[17] Volterra, V. 1937. "Principes de Biologie Mathematique." Acta Biotheoretica 3, 1-36.

[18] Whittaker ET (1999) *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (CUP, Cambridge)