Inverse scattering transform for the vector NLS equation with non-vanishing boundary conditions. II

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Outline

IST: a nonlinear Fourier Transform

A number of nonlinear evolution equations are “linearized” via the IST, i.e. associated with a pair of linear problems (Lax pair) such that the given equation results as the compatibility condition between them.

We say that the operator pair $\mathbf{X}, \mathbf{T}$ is a Lax pair for the nonlinear equation

$$q_t = F[x, t, q, q_x, q_{xx}, \dots] \qquad q = q(x, t),$$

if

$$v_x = \mathbf{X}v, \qquad v_t = \mathbf{T}v$$

[$\mathbf{X}, \mathbf{T}$ are matrix functions of $q, q_x, q_{xx}, \dots$] and the compatibility [i.e., equality of mixed derivatives $v_{xt} = v_{tx}$] is identically satisfied provided q solves the nonlinear PDE.

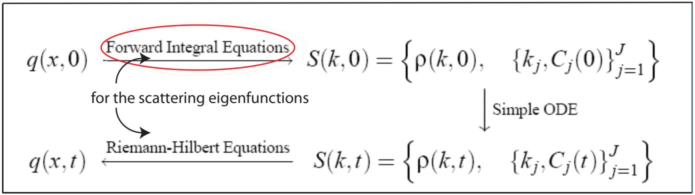
The solution of the Cauchy problem by IST proceeds in three steps, as follows:

1. the direct problem - the transformation of the initial data from the original “physical” variables ($q(x,0)$) to the transformed “scattering” variables ($S(k,0)$);
2. time dependence - the evolution of the transformed data often according to simple, explicitly solvable evolution equations (i.e., finding $S(k,t)$);
3. the inverse problem - the recovery of the evolved solution ($q(x,t)$) from the evolved solution in the transformed variables ($S(k,t)$).

Both the direct and the inverse problem make use of the first operator in the Lax pair, so-called scattering problem.

The time evolution is determined by the second operator in the Lax pair.

Schematically:



The eigenfunctions of the scattering problem play a crucial role.

Direct problem:

integral equations for the efs $\Rightarrow$ scattering data

Inverse problem:

RH equations for the efs $\Rightarrow$ reconstruction of potential

Analitycity of the eigenfunctions in k is a key issue.

Scalar Nonlinear Schrödinger (NLS) equation

The IST for the scalar defocusing NLS equation

$$iq_t = q_{xx} - 2|q|^2 q$$

with NBCs was first studied by Zakharov and Shabat in 1973.

Subsequently generalized by: Kulish et al [1976], Gerdjikov and Kulish [1978], Leon [1980], Boiti and Pempinelli [1982], Asano and Kato [1984] etc.

For an extensive study: Faddeev and Takhtajan [1987].

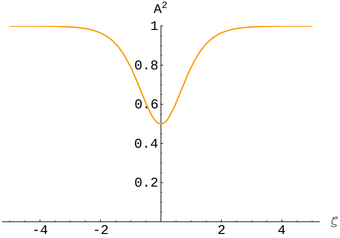
Defocusing NLS admits soliton solutions on a background

$$q(x,t) = q_0 e^{2iq_0^2 t}\left[\cos\alpha + i\sin\alpha\tanh\left(q_0\sin\alpha(x - 2q_0\cos\alpha t - x_0)\right)\right]$$

with

$$q(x,t) \to q_\pm(t) = q_0 e^{2iq_0^2 t} e^{\pm i\alpha} \qquad \text{as} \quad x \to \pm\infty$$

A gray soliton appears as a localized intensity dip of amplitude $q_0|\cos\alpha|$ on the background field q_0.



A gray soliton: A^2 is the square modulus of the solution, ζ is the coordinate in the moving frame

When the minimum amplitude is zero, i.e. $\cos\alpha = 0$, then the solution, which in this case is stationary, is referred to as a dark soliton.

Vector Nonlinear Schrödinger equation (VNLS)

The vector nonlinear Schrödinger (VNLS) equation is given by

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\,\|\mathbf{q}\|^2\,\mathbf{q}$$

where now $\mathbf{q}(x,t)$ is an N-component vector and $\|.\|$ the Euclidean norm.

IST for VNLS with NBCs not yet completely developed.

Some results in: Gerdjikov and Kulish [1985].

More recently, direct methods have been applied to the VNLS to derive explicit solutions:

Kivshar and Turitsyn [1993], Radhakrishnan and Lakshmanan [1995], Sheppard and Kivshar [1997], Nakkeeran [2001] etc.

IST for the 2-component case: BP, Ablowitz, Biondini [2006].

Atanasov and Gerdjikov [2008]: multicomponent VNLS related to certain symmetric spaces.

IST for VNLS: general ideas

The aim is to construct the IST for VNLS with NBCs for an arbitrary number of components:

N-component VNLS $\Leftrightarrow$ $N+1$ dimensional scattering problem

When $N=1$ the problem with NBCs is complicated by the fact that the scattering parameter k “lives” on a two-sheeted Riemann surface; however one still has two complete sets of analytic scattering functions.

When $N>1$, an additional complication arises: $2(N-1)$ out of the $2(N+1)$ scattering eigenfunctions are not analytic, and one has to suitably complete the basis in order to formulate a meaningful inverse problem.

$N=2$ is the case treated by ABP in 2006 and illustrated below.

When $N\geq 3$ yet another complication is added: the eigenvalue associated to the nonanalytic scattering eigenfunctions becomes a multiple eigenvalue, with multiplicity $N-1\geq 2$.

IST for VNLS: 2-component case

The direct scattering problem is first formulated on a two-sheeted covering of the complex plane.

Two out of the six Jost eigenfunctions do not admit an analytic extension on either sheet of the Riemann surface $\Rightarrow$ a suitable modification of both the direct and the inverse problem formulations is necessary.

1. On the direct side, we construct two additional analytic eigenfunctions using the adjoint problem.
2. The inverse scattering is formulated in terms of a generalized Riemann-Hilbert (RH) problem in the upper/lower half planes of a suitable uniformization variable.
3. Special soliton solutions are constructed from the poles in the RH problem (dark-dark and dark-bright solitons) that correspond to the solutions obtained with direct methods.

Lax pair

The 2-component VNLS equation

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\,\|\mathbf{q}\|^2\,\mathbf{q} \tag{1}$$

$\mathbf{q} = (q^{(1)}(x,t), q^{(2)}(x,t))^T$ is associated to the Lax pair:

$$\partial_x v = (ik\mathbf{J} + \mathbf{Q})\,v \tag{2a}$$

$$\partial_t v = \begin{pmatrix} 2ik^2 + i\mathbf{q}^T\mathbf{r} & -2k\mathbf{q}^T - i\mathbf{q}_x^T \\ -2k\mathbf{r} + i\mathbf{r}_x & -2ik^2\mathbf{I} - i\mathbf{r}\mathbf{q}^T \end{pmatrix} v \tag{2b}$$

- v is a 3-component vector
- k is the so-called scattering (or spectral) parameter

$$\mathbf{J} = \mathrm{diag}(-1,1,1), \qquad \mathbf{Q} = \begin{pmatrix} 0 & \mathbf{q}^T \\ \mathbf{r} & \mathbf{0} \end{pmatrix} \equiv \begin{pmatrix} 0 & q^{(1)}(x) & q^{(2)}(x) \\ r^{(1)}(x) & 0 & 0 \\ r^{(2)}(x) & 0 & 0 \end{pmatrix}$$

The compatibility of the Lax pair, i.e. the equality of the mixed derivatives of the 3-component vector v with respect to x and t, is equivalent to the statement that $\mathbf{q}$ satisfies the VNLS equation with $\mathbf{r} = \mathbf{q}^*$.

We consider potentials with boundary conditions as $x \to \pm\infty$ of the form

$$q^{(j)}(x,t) \sim q_\pm^{(j)}(t) = q_0^{(j)} e^{i\theta_j^\pm(t)}$$
$$r^{(j)}(x,t) \sim r_\pm^{(j)}(t) = q_0^{(j)} e^{-i\theta_j^\pm(t)}$$

i.e., with the same t-independent amplitudes at both space infinities.

Eigenfunctions

Eigenfunctions for the scattering problem are introduced by fixing the large asymptotics as $x \to \pm\infty$

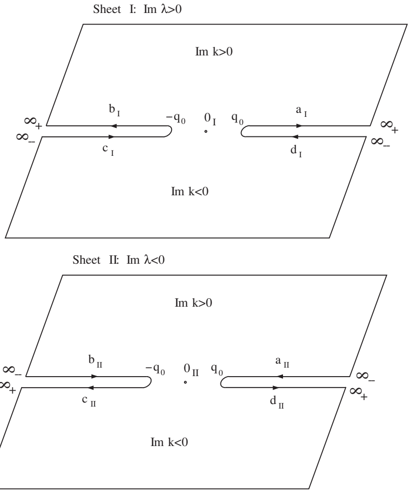
$$\phi_1(x,k) \sim w_1^- e^{-i\lambda x}, \quad \phi_2(x,k) \sim w_2^- e^{ikx}, \quad \phi_3(x,k) \sim w_3^- e^{i\lambda x}$$
$$\psi_1(x,k) \sim w_1^+ e^{-i\lambda x}, \quad \psi_2(x,k) \sim w_2^+ e^{ikx}, \quad \psi_3(x,k) \sim w_3^+ e^{i\lambda x}$$

where

$$\lambda = \sqrt{k^2 - q_0^2}, \qquad q_0^2 \equiv \|\mathbf{q}_0\|^2 = |q_\pm^{(1)}|^2 + |q_\pm^{(2)}|^2$$

$$w_1^\pm = \begin{pmatrix} \lambda + k \\ ir_\pm^{(1)} \\ ir_\pm^{(2)} \end{pmatrix}, \quad w_2^\pm = \begin{pmatrix} 0 \\ -iq_\pm^{(2)} \\ iq_\pm^{(1)} \end{pmatrix}, \quad w_3^\pm = \begin{pmatrix} \lambda - k \\ -ir_\pm^{(1)} \\ -ir_\pm^{(2)} \end{pmatrix}$$

IST for VNLS: 2-component case



$\hat{\mathbb{C}}$: Riemann surface of $\lambda^2 = k^2 - q_0^2$

Σ: cut on the Riemann surface

The triads of vectors $[\phi_1, \phi_2, \phi_3]$ and $[\psi_1, \psi_2, \psi_3]$ are each a set of linearly independent solutions of the third order scattering problem.

One can show that the eigenfunctions

ϕ_1 and ψ_3 are analytic on the upper sheet

ϕ_3 and ψ_1 are analytic on the lower sheet

ϕ_2 and ψ_2 neither

Analyticity is crucially related to the possibility of expressing the eigenfunctions in terms of meaningful Volterra integral equations.

Problem: how to substitute the non-analytic eigenfunctions?

Generalize the approach introduced by Kaup [1976] for the three-wave interaction, where the key idea was to consider, together with the given scattering problem

$$\partial_x v = (ik\mathbf{J} + \mathbf{Q})\, v \tag{3a}$$

the adjoint eigenvalue problem

$$\partial_x v^{\text{ad}} = \left(-ik\mathbf{J} + \mathbf{Q}^T\right) v^{\text{ad}} \tag{3b}$$

and use that if u^{ad}, w^{ad} are two arbitrary solutions of (3b), then

$$v = -\mathbf{J}\left(u^{\text{ad}} \wedge w^{\text{ad}}\right) e^{ikx} \tag{4}$$

is a solution of the original scattering problem (3a).

From the adjoint states, we can now construct two new solutions which by construction are analytic in the lower and upper sheet respectively.

Scattering coefficients

Introduce the scattering coefficients a_{ij} expressing one set of eigenfunctions as a linear combination of the other one

$$\phi_j(x,k) = \sum_{i=1}^{3} a_{ji}(k)\psi_i(x,k), \qquad j = 1,2,3 \tag{5a}$$

or, equivalently,

$$\psi_j(x,k) = \sum_{i=1}^{3} b_{ji}(k)\phi_i(x,k), \qquad j = 1,2,3 \tag{5b}$$

where (b_{ij}) is the inverse matrix of (a_{ij}) (both are unitary.)

The zeros of the coefficients $a_{11}(k)$ and $a_{33}(k)$ [or $b_{11}(k)$ and $b_{33}(k)$] provide the discrete eigenvalues of the scattering problem.

Uniformization

Following an idea proposed by Faddeev and Takhtajan [1987] for the scalar NLS, we introduce a uniformization variable z defined by the conformal mapping

$$z = z(k) = k + \lambda(k) \tag{6}$$

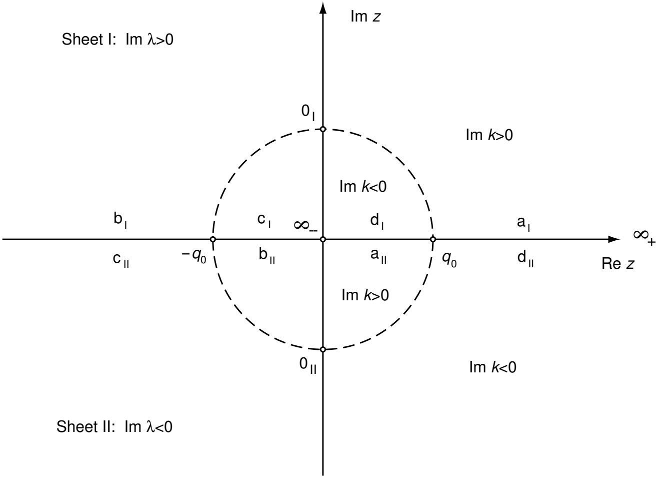
The inverse mapping is given by

$$k = k(z) = \frac{1}{2}\left(z + \frac{q_0^2}{z}\right) \tag{7a}$$

$$\lambda(k) = z - k = \frac{1}{2}\left(z - \frac{q_0^2}{z}\right) \tag{7b}$$

This uniformization allows to formulate the RH equations for the inverse problem on the complex plane rather then on a Riemann surface.

1. Cuts on the Riemann surface $\rightarrow$ real z axis
2. Sheets $\mathbb{C}_1/\mathbb{C}_2 \rightarrow$ upper/lower half planes of z
3. $(-q_0, q_0)$ on each sheet $\rightarrow$ upper/lower semicircles of radius q_0



For the scalar NLS:

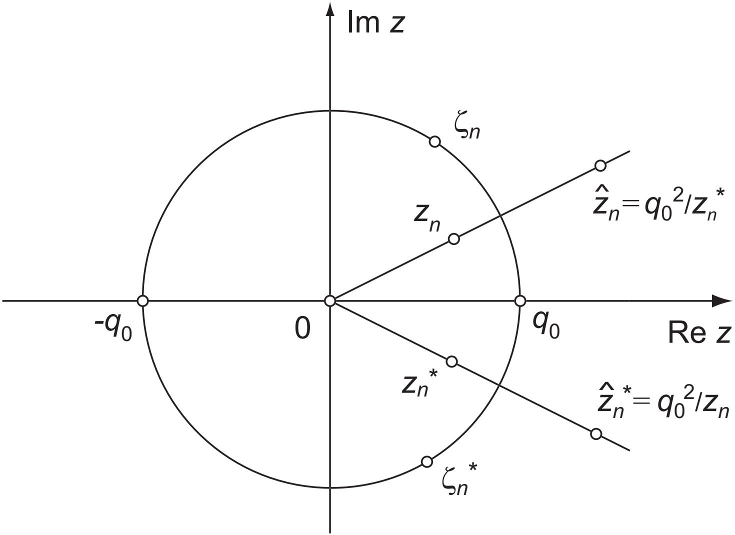
Symmetries + self-adjointness + unitarity of the scattering matrix confine the eigenvalues on the circle $|z| = q_0$

For the vector NLS:

One can have:

- pairs of eigenvalues $\{\zeta_n, \zeta_n^*\}$ on the circle of radius q_0
- quartets of eigenvalues $\{z_n, z_n^*, q_0^2/z_n, q_0^2/z_n^*\}$ off the circle

IST for VNLS: 2-component case



Location of the discrete eigenvalues in the complex z-plane

Explicit solutions: dark-dark and dark-bright solitons

In the case of one pair of eigenvalues ζ_1, ζ_1^* on the circle of radius q_0 one gets a dark-dark soliton solution

$$\mathbf{q}(x,t) = \mathbf{q}_+(0)e^{2iq_0^2 t}\left\{\cos\alpha + i\sin\alpha \tanh[\nu_1(x-2k_1 t)]\right\} e^{2iq_0^2 t}$$

$$\zeta_1 = k_1 + i\nu_1 = q_0 e^{-i\alpha}$$

With one quartet of eigenvalues off the circle of radius q_0 (with principal eigenvalue $z_1 = k_1 + i\nu_1$),

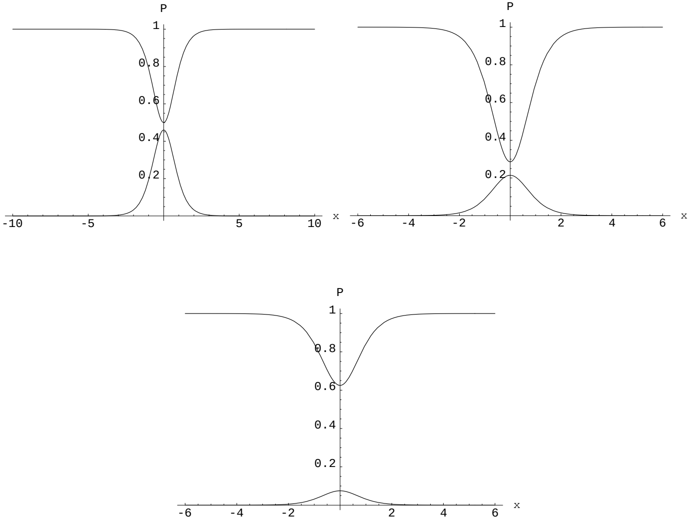
$$q^{(1)}(x,t) = -\nu_1 \sin\alpha \sqrt{q_0^2 - |z_1|^2}\,\mathrm{sech}[\nu_1(x-2k_1 t)]\, e^{-ik_1 x + i[2q_0^2 + (k_1^2 - \nu_1^2)]t}$$

$$q^{(2)}(x,t) = q_0\left\{\cos\alpha + i\sin\alpha \tanh[\nu_1(x-2k_1 t)]\right\} e^{2iq_0^2 t}$$

where

$$k_1 = |z_1|\cos\alpha\,, \qquad \nu_1 = -|z_1|\sin\alpha\,.$$

IST for VNLS: 2-component case



Amplitudes of the “dark” and “bright” components for several values of k_1, ν_1.

N component VNLS

The scattering problem for N-component defocusing VNLS is given by the following $N+1$-dimensional system

$$v_x = \mathbf{L}v, \qquad \mathbf{L} = ik\mathbf{J} + \mathbf{Q}$$

$$\mathbf{J} = \mathrm{diag}\left(-1, \underbrace{1, \dots, 1}_{N}\right) \qquad \mathbf{Q} = \begin{pmatrix} 0 & \mathbf{q}^T \\ \mathbf{r} & \mathbf{0}_N \end{pmatrix}$$

$$\mathbf{q}^T = \left(q^{(1)}, \dots, q^{(N)}\right) \qquad \mathbf{r} = \mathbf{q}^*$$

with asymptotic behavior

$$\lim_{x\to\pm\infty} \mathbf{Q}(x) = \mathbf{Q}_\pm\,, \qquad \mathbf{q}_\pm^T = \left(q_\pm^{(1)}, \dots, q_\pm^{(N)}\right)$$

$$q_\pm^{(j)} = q_{\pm,0}^{(j)} e^{i\theta_\pm^{(j)}}\,, \qquad ||\mathbf{q}_+|| = ||\mathbf{q}_-|| = q_0$$

Eigenvalues are given by $\pm i\lambda, \underbrace{ik, \dots, ik}_{N-1}$ where $\lambda = \sqrt{k^2 - q_0^2}$

and the corresponding eigenvectors are

$$\{-i\lambda, ik, \dots, ik, i\lambda\} \leftrightarrow \mathbf{\Lambda}^{\pm} = \begin{pmatrix} \lambda + k & \mathbf{0}_{1\times(N-1)} & \lambda - k \\ i\mathbf{r}_{\pm} & i\mathbf{Q}_{\pm}^{\perp} & -i\mathbf{r}_{\pm} \end{pmatrix}.$$

$\mathbf{Q}_{\pm}^{\perp}$ has $N-1$ columns orthogonal to $\mathbf{q}_{\pm}$.

The ordering of the eigenvalues plays a crucial role. Since

$$\begin{aligned} \operatorname{Im}\lambda, \operatorname{Im}(\lambda \pm k) \geq 0 \qquad &\text{on } \mathbb{C}_1 \\ \operatorname{Im}\lambda, \operatorname{Im}(\lambda \pm k) \leq 0 \qquad &\text{on } \mathbb{C}_2 \end{aligned}$$

we write the matrix of eigenvalues as

$$\mathbf{J}_k = \operatorname{diag}(\alpha_1, \dots, \alpha_{N+1})$$

and $\alpha_j = k$ for $2 \leq j \leq N$ while $\alpha_1 = -\lambda$, $\alpha_{N+1} = \lambda$ on $\mathbb{C}_1$ and vice-versa on $\mathbb{C}_2$.

It is convenient to rewrite the scattering problem by expressing Φ and Ψ in the form

$$\Phi(x,k) = m(x,k)e^{ix\mathbf{J}_k}, \qquad \Psi(x,k) = \tilde{m}(x,k)e^{ix\mathbf{J}_k}.$$

[remember: $\mathbf{J}_k = \mathrm{diag}(\alpha_1, \ldots, \alpha_{N+1})$]
Then the problem becomes: given $k \in \hat{\mathbb{C}}/\Sigma$, determine matrices m and $\tilde{m}$ solutions of the appropriate differential equations with BCs

$$\begin{cases} \lim_{x\to-\infty} m(x,k) = \mathbf{\Lambda}^- \\ m(\cdot,k) \text{ is bounded} \end{cases} \tag{8a}$$

and

$$\begin{cases} \lim_{x\to+\infty} \tilde{m}(x,k) = \mathbf{\Lambda}^+ \\ \tilde{m}(\cdot,k) \text{ is bounded} \end{cases} \tag{8b}$$

Fundamental matrices

Definition

A fundamental matrix for the operator **L** and the point $k \in \hat{\mathbb{C}} \setminus \Sigma$ is either a solution $m(\cdot, k)$ of (8a) or a solution $\tilde{m}(\cdot, k)$ of (8b).

Let $e_1, \ldots, e_{N+1}$ denote the standard basis vectors for $\mathbb{C}^{N+1}$, considered as a space of column vectors. Then $m_j = me_j$ and $\tilde{m}_j = \tilde{m}e_j$ are the j-th columns of the matrices m and $\tilde{m}$.

The ordering chosen for the eigenvalues implies that the columns m_1 and $\tilde{m}_{N+1}$ can be obtained as solutions of Volterra equations.

In order to complete the basis of analytic eigenfunctions, we generalize the approach suggested by Beals, Deift and Tomei [1988] for general scattering and inverse scattering on the line, but with nonvanishing BCs.

Fundamental tensors

Consider the wedge products with values in the exterior algebra $\Lambda(\mathbb{C}^{N+1}) = \oplus \Lambda^j\left(\mathbb{C}^{N+1}\right)$

$$f_j(\cdot,k) = m_1(\cdot,k) \wedge m_2(\cdot,k) \wedge \cdots \wedge m_j(\cdot,k) \tag{9a}$$

$$g_j(\cdot,k) = \tilde{m}_j(\cdot,k) \wedge \tilde{m}_{j+1}(\cdot,k) \wedge \cdots \wedge \tilde{m}_{N+1}(\cdot,k)\,. \tag{9b}$$

The equations for m_j and $\tilde{m}_j$ imply that f_j, g_j satisfy appropriate differential equations, with asymptotic conditions

$$\lim_{x\to-\infty} f_j(x,k) = \mathbf{\Lambda}^- e_1 \wedge \cdots \wedge \mathbf{\Lambda}^- e_j \tag{10a}$$

$$\lim_{x\to+\infty} g_j(x,z) = \mathbf{\Lambda}^+ e_j \wedge \cdots \wedge \mathbf{\Lambda}^+ e_{N+1} \tag{10b}$$

Definition

A fundamental tensor family for the operator $\mathbf{L}$ and $k \in \hat{\mathbb{C}} \setminus \Sigma$ is a set of solutions $\{f_j, g_j\}_{j=1}^{N+1}$ to the above problem.

Theorem

*For each $k \in \hat{\mathbb{C}} \backslash \Sigma$ there is a unique fundamental tensor family for $\mathbf{L}$ and k. On each component of $\mathbb{C}_1 \backslash \Sigma$ and $\mathbb{C}_2 \backslash \Sigma$ these families are **analytic** functions of k. They extend smoothly to Σ (with the exception, possibly, of the branch points $\pm q_0$) and these extensions satisfy the boundary conditions as well.*

The idea of the proof is simply to show that, like $f_1 = m_1$ and $g_{N+1} = \tilde{m}_{N+1}$, all the f_j and g_j are solutions of Volterra equations and then proceed by Neumann series iteration.

Then the first step in the direct problem will now be: solve the Volterra integral equations to find the (unique) fundamental tensor family $\{f_j, g_j\}_{j=1}^{N+1}$.

IST for VNLS: N components

Given the fundamental tensor family $\{f_j, g_j\}_{j=1}^{N+1}$, one can then define scalar functions $\Delta_j(k)$, $0 \leq j \leq N+1$, analytic on $\hat{\mathbb{C}}/\Sigma$, and having smooth extensions on $\hat{\mathbb{C}}/\{\pm q_0\}$, such that on $\hat{\mathbb{C}}/\Sigma$

$$\lim_{x\to+\infty} f_j(x,k) = \Delta_j(k)\left[\mathbf{\Lambda}^- e_1 \wedge \cdots \wedge \mathbf{\Lambda}^- e_j\right] \quad (11a)$$

$$\lim_{x\to-\infty} g_{j+1}(x,k) = \Delta_j(k)\left[\mathbf{\Lambda}^+ e_{j+1} \wedge \cdots \wedge \mathbf{\Lambda}^+ e_{N+1}\right] \quad (11b)$$

For consistency one obviously has:

$$\Delta_{N+1} \equiv 1, \qquad f_{N+1}(x,k) = \mathbf{\Lambda}^- e_1 \wedge \cdots \wedge \mathbf{\Lambda}^- e_{N+1} \quad (12a)$$

$$\Delta_0 \equiv 1, \qquad g_1(x,k) = \mathbf{\Lambda}^+ e_1 \wedge \cdots \wedge \mathbf{\Lambda}^+ e_{N+1} \quad (12b)$$

From fundamental tensors to fundamental matrices

The key point is, given the fundamental tensor families $\{f_j, g_j\}_{j=1}^{N+1}$, to construct the columns of the matrices m and $\tilde{m}$, i.e. the fundamental matrices.

The first step is to show that the fundamental tensors are decomposable. In fact, it is possible to prove the following.

For each $k \in \hat{\mathbb{C}} \setminus \{\pm q_0\}$, there are unique smooth functions $v_j(\cdot, k) : \mathbb{R} \to \mathbb{C}^{N+1}$ and $w_j(\cdot, k) : \mathbb{R} \to \mathbb{C}^{N+1}$ such that

$$v_1 \wedge \cdots \wedge v_j \equiv f_j, \qquad w_j \wedge \cdots \wedge w_{N+1} \equiv g_j \tag{13a}$$

and

$$\begin{cases} v_1, \dots, v_j \text{ are orthogonal} \\ w_j, \dots, w_{N+1} \text{ are orthogonal} \end{cases} . \tag{13b}$$

Then there is the crucial step in passing from the v_k to the m_k.

Theorem

Suppose $f_{j-1} \in \Lambda^{j-1}\left(\mathbb{C}^{N+1}\right), f_j \in \Lambda^j\left(\mathbb{C}^{N+1}\right)$ *and* $g_j \in \Lambda^{N-j+2}\left(\mathbb{C}^{N+1}\right)$ *are decomposable elements with*

$$f_j = f_{j-1} \wedge v_j, \qquad f_{j-1} \wedge g_j \neq 0.$$

Then there is a unique $m_j \in \mathbb{C}^{N+1}$ *such that*

$$f_{j-1} \wedge m_j = f_j, \qquad m_j \wedge g_j = 0.$$

Moreover, all entries of m_j *can be expressed locally as rational functions of the coefficients of* f_{j-1}, f_j *and* g_j *with respect to the standard basis.*

The dual result for $\tilde{m}_j$ can be stated analogously.

Suppose $k \in \hat{\mathbb{C}} \setminus \{\pm q_0\}$ and $\Delta_{j-1}(k) \neq 0$. Then there is a unique bounded function $m_j(\cdot, k) : \mathbb{R} \to \mathbb{C}^{N+1}$ that satisfies the proper differential equation and the BCs

$$\lim_{x\to-\infty} \mathbf{\Lambda}^- e_1 \wedge \cdots \wedge \mathbf{\Lambda}^- e_{j-1} \wedge m_j = \mathbf{\Lambda}^- e_1 \wedge \cdots \wedge \mathbf{\Lambda}^- e_j \tag{14a}$$

$$\lim_{x\to+\infty} m_j \wedge \mathbf{\Lambda}^- e_{j+1} \wedge \cdots \wedge \mathbf{\Lambda}^- e_{N+1} = \Delta_{j-1}^{-1} \Delta_j \mathbf{\Lambda}^- e_j \wedge \cdots \wedge \mathbf{\Lambda}^- e_{N+1} \tag{14b}$$

If $k \in \hat{\mathbb{C}}/\Sigma$ or $j = 1$ then (14a) can be replaced by the stronger assumption

$$\lim_{x\to-\infty} m_j = \mathbf{\Lambda}^- e_j \tag{15}$$

and if $k \in \hat{\mathbb{C}}/\Sigma$ or $j = 1$ then (14b) can be replaced by the stronger assumption

$$\lim_{x\to+\infty} m_j = \Delta_{j-1}(k)^{-1} \Delta_j(k) \mathbf{\Lambda}^- e_j. \tag{16}$$

Away from the zeros of Δ_{j-1} (resp. Δ_j) in $\hat{\mathbb{C}}/\{\pm q_0\}$, the function m_j (resp. $\tilde{m}_j$) depends smoothly on x and k and are analytic functions of k for $k \in \hat{\mathbb{C}}\backslash\Sigma$.

The fundamental matrices m and $\tilde{m}$ exist for each $k \in \hat{\mathbb{C}}\backslash\Sigma$, apart from a bounded, discrete set

$$Z = \left\{ k \in \hat{\mathbb{C}}\backslash\Sigma : \prod_{j=1}^{N} \Delta_j(k) = 0 \right\}$$

Moreover,

$$m(x,k) = \tilde{m}(x,k)\delta(k)$$

where

$$\delta(k) = \mathrm{diag}\left\{\Delta_1/\Delta_0, \Delta_2/\Delta_1, \ldots, \Delta_{N+1}/\Delta_N\right\}.$$

For each $x \in \mathbb{R}$, $m(x,\cdot)$ is analytic on $\hat{\mathbb{C}}/(\Sigma \cup Z)$ and has a pole of positive order at each point of Z.

We showed how to generalize the construction of FASs for defocusing VNLS with NBCs to an arbitrary number of components and introduce analytical scattering data.

This is a work in progress, still some things to be done.

In particular:

- Formulate the Inverse Problem [RH problem: the analytic properties of eigenfunctions and scattering data are now known]
- Find and characterize vector solitons
- Investigate soliton interactions