Formal Groups, Integrable Systems and Number Theory

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Gallipoli, June 18, 2008

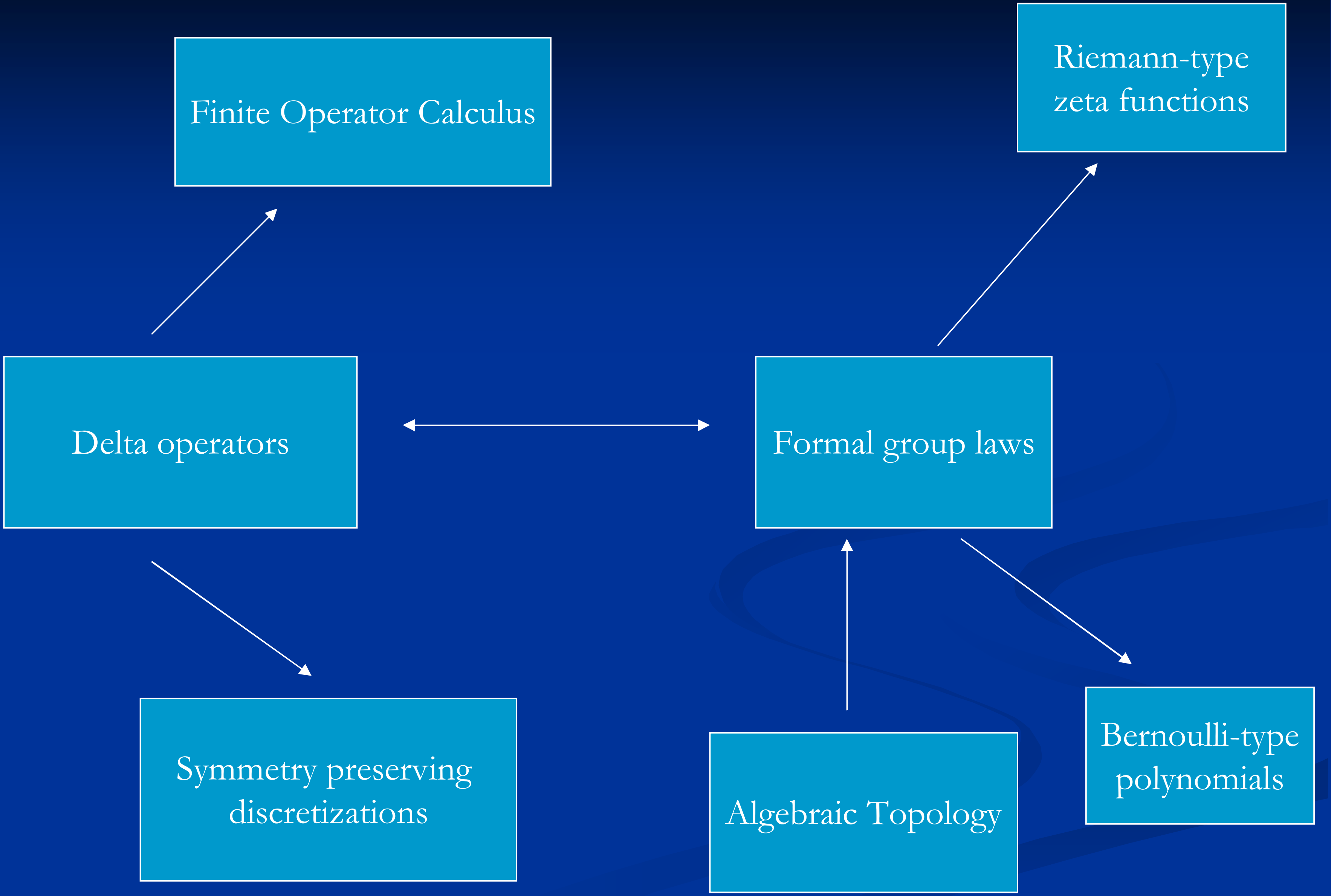
Outline: the main characters

- Formal groups: a brief introduction

- Finite operator theory
  - Symmetry preserving Discretization of PDE's
  - Formal solutions of linear difference equations

Application: Superintegrability
  - Exact (and Quasi Exact) Solvability
  - Integrals of motion

- Number Theory
  - **Generalized Riemann zeta functions**
  - New Appell polynomias of Bernoulli type



Finite Operator Calculus

Riemann-type zeta functions

Delta operators

Formal group laws

Symmetry preserving discretizations

Algebraic Topology

Bernoulli-type polynomials

- P. Tempesta, A. Turbiner, P. Winternitz, J. Math. Phys, 2002
- D. Levi, P. Tempesta. P. Winternitz, J. Math. Phys., 2004
- D. Levi, P. Tempesta. P. Winternitz, Phys Rev D, 2005
- P. Tempesta., C. Rend. Acad. Sci. Paris, 345, 2007
- P. Tempesta., J. Math. Anal. Appl. 2008
- S. Marmi, P. Tempesta, generalized Lipschitz summation formulae and hyperfunctions 2008, submitted
- P. Tempesta, L - series and Hurwitz zeta functions associated with formal groups and universal Bernoulli polynomials (2008)

Formal group laws

Let R be a commutative ring with identity

$R\{x_1, x_2, \ldots\}$ be the **ring of formal power series** with coefficients in R

Def 1 A one-dimensional **formal group law** over R is a formal power series $\Phi(x, y) \in R\{x, y\}$ s.t.

$$\Phi(x,0) = \Phi(0, x) = x$$
$$\Phi(\Phi(x, y), z) = \Phi(x, \Phi(y, z))$$

When $\Phi(x, y) = \Phi(y, x)$ the formal group is said to be commutative.

$\exists$ a unique formal series $\varphi(x) \in R\{x\}$ such that $\Phi(x, \varphi(x)) = 0$

Def 2 An n-dimensional **formal group law** over R is a collection of n formal power series $\Phi_j(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in 2n variables, such that

$$\mathbf{\Phi}(\mathbf{x},0) = \mathbf{x}$$

$$\mathbf{\Phi}(\mathbf{x}, \mathbf{\Phi}(\mathbf{y}, \mathbf{z})) = \mathbf{\Phi}(\mathbf{\Phi}(\mathbf{x}, \mathbf{y}), \mathbf{z})$$

Examples

1) The **additive** formal group law

$$\Phi(x, y) = x + y$$

2) The **multiplicative** formal group law

$$\Phi(x, y) = x + y + xy$$

3) The **hyperbolic** one (addition of velocities in special relativity)

$$\Phi(x, y) = (x + y)/(1 + xy)$$

4) The formal group for **elliptic integrals** (Euler)

$$\Phi(x, y) = \left(x\sqrt{1 - y^4} + y\sqrt{1 - x^4}\right)/\left(1 + x^2 y^2\right)$$

Indeed:

$$\int_0^x \frac{dt}{\sqrt{1 - t^4}} + \int_0^y \frac{dt}{\sqrt{1 - t^4}} = \int_0^{\Phi(x,y)} \frac{dt}{\sqrt{1 - t^4}}$$

Connection with Lie groups and algebras

- More generally, we can construct a formal group law of dimension n from any **algebraic group** or **Lie group** of the same dimension n, by taking coordinates at the identity and writing down the formal power series expansion of the product map. An important special case of this is the formal group law of an **elliptic curve** (or **abelian variety**)

- Viceversa, **given a formal group law we can construct a Lie algebra.**

  Let us write:

$$\Phi(x,y)=x+y+\Phi_2(x,y)+\Phi_3(x,y)+\dots$$

defined in terms of the quadratic part $\Phi_2(x,y)$:

Any n- dimensional formal group law gives an n dimensional Lie algebra over the ring R,

$$[\mathbf{x},\mathbf{y}]=\boldsymbol{\Phi}_2(\mathbf{x},\mathbf{y})-\boldsymbol{\Phi}_2(\mathbf{y},\mathbf{x})$$

Algebraic groups ⟶ **Formal group laws** ⟶ **Lie algebras**

- **Bochner**, 1946
- **Serre**, 1970 -
- **Novikov, Bukhstaber**, 1965 -

Def. 3. Let $c_1, c_2, \ldots$ be indeterminates over $\mathbf{Q}$ The **formal group logarithm** is

$$F(s) = s + c_1 \frac{s^2}{2} + c_2 \frac{s^3}{3} + \ldots$$

The associated **formal group exponential** is defined by

$$G(t) = t - c_1 \frac{t^2}{2} + \left(3c_1^2 - 2c_2\right)\frac{t^3}{6} + \ldots$$

so that $F(G(t)) = t$

Def 4. The formal group defined by $\Phi(s_1, s_2) = G(F(s_1) + F(s_2))$ is called

the **Lazard Universal Formal Group**

The **Lazard Ring** is the subring of $\mathbf{Q}[c_1, c_2, \ldots]$ generated by the coefficients of the power series $G(F(s_1) + F(s_2))$

- Algebraic topology: cobordism theory
- Analytic number theory
- Combinatorics

Bukhstaber, Mischenko and Novikov : All fundamental facts of the theory of unitary cobordisms, both modern and classical, can be expressed by means of Lazard's formal group.

Given a function G(t), there is always a delta difference operator with specific properties whose representative is G(t)

Main idea

- The theory of formal groups is naturally connected with finite operator theory.
- It provides an elegant approach to discretize continuous systems, in particular superintegrable systems, in a symmetry preserving way
- Such discretizations correspond with a class of interesting number theoretical structures (Appell polynomials of Bernoulli type, zeta functions), related to the theory of formal groups.

Introduction to finite operator theory

Silvester, Cayley, Boole, Heaviside, Bell,..   Umbral Calculus

$$Dx^n = nx^{n-1} \qquad \Delta(x)_n = n(x)_{n-1} \qquad (x)_n = x(x-1)\ldots(x-n+1)$$

$$(x+a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k x^{n-k} \qquad (x+a)_n = \sum_{k=0}^{\infty} \binom{n}{k} (a)_k (x)_{n-k}$$

G. C. Rota and coll., M.I.T., 1965-

- Di Bucchianico, Loeb (Electr.J. Comb., 2001, survey)

$\mathrm{F} \equiv \mathrm{F}[[t]]$, $\mathrm{P} \equiv \mathrm{P}[t]$ ${}_s\ f \in \mathrm{F} \longrightarrow f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k$

$$\langle f(t) | x^n \rangle = a_n \qquad \langle t^k | x^n \rangle = n!\,\delta_{n,k}$$

$$t^k x^n = \begin{cases} (n)_k x^{n-k}, & k \le n \\ 0, & k > n \end{cases} \qquad f(t)x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}$$

F — algebra of f.p.s.

F — algebra isomorphic to P*

F — subalgebra of L (P)

F_s : subalgebra of shift-invariant operators

$$Tf(x) = f(x+\sigma)$$
$$[S,T] = 0$$

$p_n(x)$ polynomial in x of degree n.

Def 5. $Q \in \mathrm{F}_s$ is a delta operator if $Q\,x = c \neq 0$.

Def 6. $\{p_n(x)\}_{n\in\mathbf{N}}$ is a sequence of basic polynomials for Q if

$$Qp_n(x) = np_{n-1}(x)$$

$$p_0(x) = 1 \qquad\qquad p_n(0) = 0 \quad \forall n$$

$$\begin{tikzcd} Q \in \mathrm{F}_s \arrow[r, leftrightarrow] & \{p_n(x)\}_{n\in\mathbf{N}} \end{tikzcd}$$

Def 7. An umbral operator R is an operator mapping basic sequences into basic sequences:

$$\{p_n(x)\}_{n\in\mathbf{N}}\Big|_{Q_1} = \{q_n(x)\}_{n\in\mathbf{N}}\Big|_{Q_2}$$

Finite operator theory and Algebraic Topology

E: complex orientable spectrum $\qquad \Delta^E = D + c_1\dfrac{D^2}{2!} + \ldots + c_{i-1}\dfrac{D^i}{i!} + \ldots$

Appell polynomials

$$\{a_n(x)\}_{n\in N} \qquad \partial_x a_n(x) = na_{n-1}(x) \qquad a_0(x) = c \neq 0$$

Additional structure in F_s: **Heisenberg-Weyl** algebra

D. Levi, P. T. and P. Winternitz, J. Math. Phys. 2004,
D. Levi, P. T. and P. Winternitz, Phys. Rev. D, 2004

Q: delta operator, $\beta \in F_s$, $[Q, x\beta] = 1$

Lemma a) $\beta = (Q')^{-1}$, $Q' = [Q, x]$

b) $\left[Q, (x\beta)^{\alpha}\right] = \alpha (x\beta)^{\alpha - 1}$

$\left\{(x\beta)^n\right\}_{n \in N}$: basic sequence of operators for Q

R: $L(P) \longrightarrow L(P)$

$$\begin{tikzcd}
\left\{(x\beta_1)^n\right\}_{n\in N} \arrow[r, leftrightarrow, "R"] \arrow[d, leftrightarrow] & \left\{(x\beta_2)^n\right\}_{n\in N} \arrow[d, leftrightarrow] \\
Q_1 \arrow[r, leftrightarrow, "R"] & Q_2
\end{tikzcd}$$

$$\begin{tikzcd}
\left\{x^n\right\}_{n\in N} \arrow[r, leftrightarrow, "R"] \arrow[d, leftrightarrow] & \left\{(x\beta)^n\right\}_{n\in N} \arrow[d, leftrightarrow] \\
\partial_x \arrow[r, leftrightarrow, "R"] & \Delta
\end{tikzcd}$$

Delta operators, formal groups and basic sequences

$$\Delta_q = \frac{1}{\sigma}\sum_{k=l}^{m} a_k T^k \quad l,m \in Z \qquad \sum_k a_k = 0 \qquad \sum_k k a_k = 1 \qquad [\Delta, x\beta] = 1$$

(Formal group exponentials)

Simplest example: $Q = \partial_x \quad \beta = 1 \quad p_n = x^n$

Discrete derivatives:

$$Q = \Delta^+ = \frac{T-1}{\sigma} \qquad \beta = T^{-1} \qquad p_n(x) = (x)_n = x(x-\sigma)\ldots(x-(n-1)\sigma)$$

Theorem 1: The sequence of polynomials $P_n(x) \equiv x_n^{[q]} = (x\beta)^n \cdot 1$ satisfies:

$$\Delta_q x_n^{[q]} = n x_{n-1}^{[q]} \qquad p_0(x) = 1 \qquad p_n(0) = 0$$

$$x_n^{[q]} = \sum_{k=0}^{n} s^{[q]}(n,k) x^k \qquad x^n = \sum_{k=0}^{n} S^{[q]}(n,k) x_k^{[q]} \qquad \sum_{n=k}^{\infty} S^{[q]}(n,k)\frac{t^n}{n!} = \frac{(\Delta_q)^k}{k!}$$

$s^{[q]}(n,k)$ **generalized Stirling numbers of first kind**

$S^q(n,k)$ **generalized Stirling numbers of second kind**

$$(x+y)_{[q]}^N = \sum_{k=0}^{N} \binom{n}{k} x_k^{[q]} y_{n-k}^{[q]} \qquad \forall q \in \qquad \text{Appell property)}$$

$$\sum_k s^{[q]}(n,k) S^{[q]}(k,m) = \sum_k S^{[q]}(n,k) s^{[q]}(k,m) = \delta_{m,n}$$

Finite operator theory and Lie Symmetries

$$E_a(x,u,u_x,u_{xx},\ldots,u_{nx})=0, \quad x\in R^p, u\in R^q, a=1,\ldots,s$$

$\hat{X}$ generator of a symmetry group $\hat{X} = \sum_{i=1}^{p} \xi_i(x,u)\partial_{x_i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u)\partial_{u_\alpha}$

- Invariance condition (Lie's theorem):

$$pr^{(n)}\hat{X}E_a\big|_{E_1=\ldots E_s=0}=0, \quad a=1,\ldots,s$$

I) Generate classes of exact solutions from known ones.

II) Perform Symmetry Reduction:

a) reduce the number of variables in a *PDE* and obtain particular solutions satisfying certain boundary conditions: group invariant solutions.

b) reduce the order of an ODE.

III) Identify equations with isomorphic symmetry groups.
They may be transformed into each other.

Many kinds of continuous symmetries are known:

- Classical Lie-point symmetries
  - group invariant sol.
  - part. invariant sol.
- Higher-order symmetries
  - contact symmetries
  - generalized symmetries
  - master symmetries
- Nonclassical symmetries
  - conditional symmetries
  - partial symmetries
  - λ symmetries
- Approximate symmetries
- Nonlocal symmetries (potential symmetries, theory of coverings, WE prolongation structures, pseudopotentials, ghost symmetries…)

etc. (A. Grundland, P. T. and P. Winternitz, J. Math. Phys. (2003))

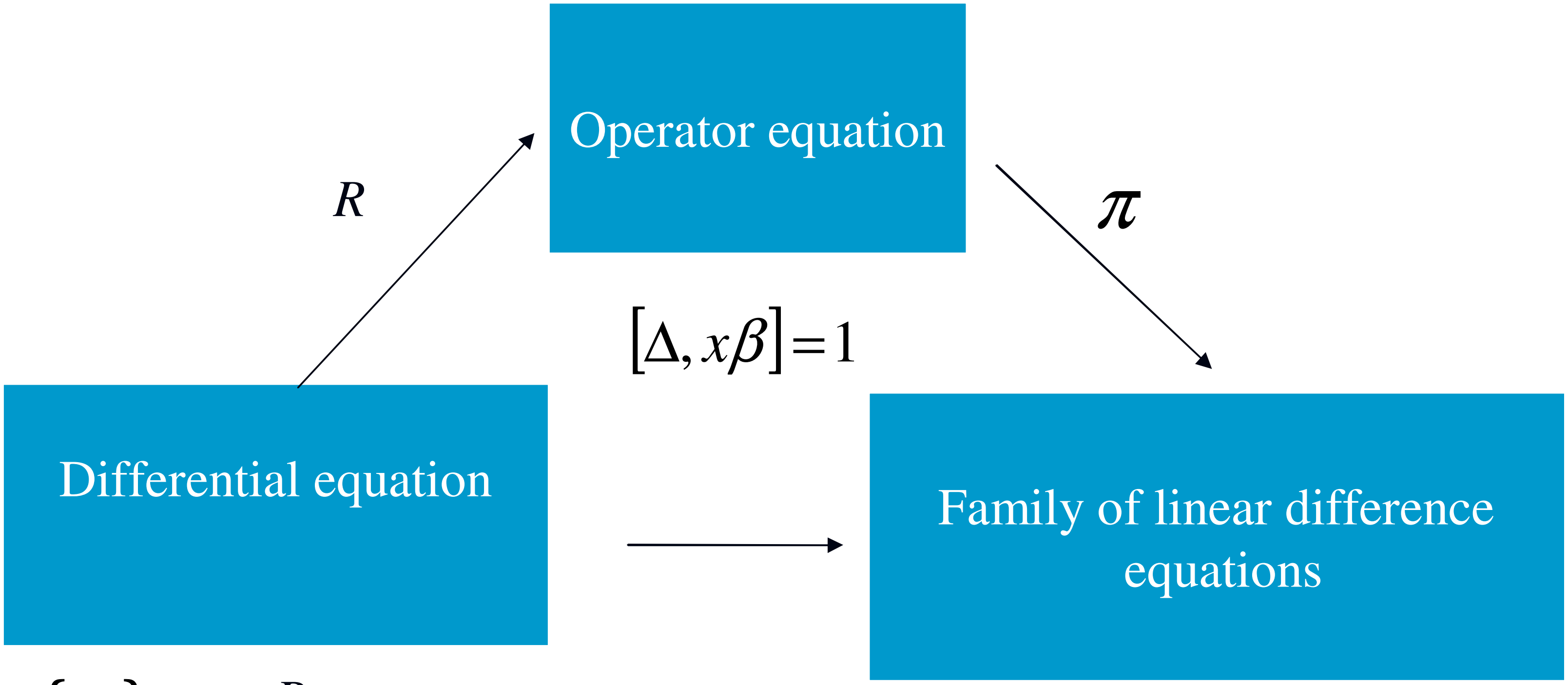
Problems: **how to extend the theory of Lie symmetries to Difference Equations?**

how to discretize a differential equation in such a way that its symmetr properties are preserved?

Generalized point symmetries of Linear Difference Equations

- D. Levi, P. T. and P. Winternitz, JMP, 2004

Reduce to classical point symmetries in the continuum limit.



Operator equation

R

π

$[\Delta, x\beta] = 1$

Differential equation

Family of linear difference equations

$$\begin{tikzcd}
\{x^n\}_{n\in N} \arrow[r, leftrightarrow, "R"] \arrow[d, leftrightarrow] & \{(x\beta)^n\}_{n\in N} \arrow[d, leftrightarrow] \\
\partial_x \arrow[r, leftrightarrow, "R"] & \Delta
\end{tikzcd}$$

$$R : E(x,u) \rightarrow \tilde{E}(x\beta,\tilde{u})$$

$$\pi : \quad (x\beta)^n \rightarrow P_n(x)$$

Theorem 2

Let E be a linear PDE of order n $\geq$ 2 or a linear ODE of order n ≥ 3 with constants or polynomial coefficients and $\tilde{E} = R\,E$ be the corresponding operator equation. All difference equations obtained by specializing and projecting $\tilde{E}$ possess a subalgebra of Lie point or higher-order symmetries isomorphic to the Lie algebra of symmetries of *E*.

- Differential equation $\sum_{k=0}^{n} c_k \partial_x{}^k f(x) = 0$
- Operator equation $\sum_{k=0}^{n} c_k Q^k f(x\beta) = 0$

Family of difference equations

$$Q \equiv \Delta_q \qquad \sum_{k=0}^{n} c_k \Delta_q{}^k F(x) = 0 \qquad F(x) = f(x\beta)\cdot 1 = f(P_n(x))$$

$$\{P_n(x)\}_{n\in N} \quad : \text{basic sequence for } \Delta_q$$

Consequence: two classes of symmetries for linear $P\Delta Es$

Generalized point symmetries $\xleftrightarrow{\;R\;}$ Isom. to cont. symm.

Purely discrete symmetries $\longleftrightarrow$ No continuum limit

Superintegrable Systems in Quantum Mechanics

- Classical mechanics — Symplectic manifold (M, ω)
- Integral of motion: $\{H, F\} = 0 \qquad \frac{\partial F}{\partial t} = 0$
- Quantum mechanics — Hilbert space: $L^2(R^n, \mu)$
- Integral of motion: $[H, X] = 0 \qquad \frac{\partial X}{\partial t} = 0$

A system is said to be

- Integrable $I = n$
- Superintegrable $I > n$

- minimally superintegrable if $I = n + 1$
- maximally superintegrable if $I = 2n - 1$

Stationary Schroedinger equation (in E_2)

$$H\psi = E\psi \qquad H = -\frac{1}{2}\nabla^2 + V(x, y)$$

Superintegrability

Generalized symmetries

Exact solvability

- M.B. Sheftel, P. T. and P. Winternitz, J. Math. Phys. (2001)
- A. Turbiner, P. T. and P. Winternitz, J. Math. Phys (2001).

There are four superintegrable potentials admitting two integrals of motion which are second order polynomials in the momenta:

$$V_I = \omega^2\left(x^2 + y^2\right) + \frac{a}{x^2} + \frac{b}{y^2} \qquad V_{II} = \omega^2\left(4x^2 + y^2\right) + \frac{a}{y^2} + bx$$

$$V_{III} = \frac{a}{r} + \frac{1}{r^2}\left(\frac{b + c\cos\vartheta}{\sin^2\vartheta}\right) \qquad V_{IV} = \frac{2a + b\xi + c\eta}{\xi^2 + \eta^2}$$

Smorodinski-Winternitz potentials

They are superseparable

General structure of the integrals of motion

$$X = aL_3^{\ 2} + b\left(L_3P_1 + P_1L_3\right) + c\left(L_3P_2 + P_2L_3\right) + d\left(P_1^{\ 2} - P_2^{\ 2}\right)$$
$$+ 2eP_1P_2 + \alpha L_3 + \beta P_2 + \phi(x, y)$$

$$\left[H, X\right] = 0$$

with

$$P_1 = \partial_x \qquad P_2 = \partial_y \qquad L_3 = y\partial_x - x\partial_y$$

The umbral correspondence immediately provides us with discrete versions of these systems.

$$H_I^{\ D} = -\frac{1}{2}\left(\Delta_x^{\ 2} + \Delta_y^{\ 2}\right) + \frac{\omega^2}{2}\left[(x\beta_x)^2 + \left(y\beta_y\right)^2\right] + \frac{a}{2}(x\beta_x)^{-2} + \frac{b}{2}(y\beta y)^2$$

$$X_1^{\ D} = \left[-\frac{1}{2}\Delta_x^{\ 2} + \omega^2(x\beta_x)^2 + a(x\beta_x)^{-2}\right] - \left[-\frac{1}{2}\Delta_y^{\ 2} + \omega^2\left(y\beta_y\right)^2 + b\left(y\beta_y\right)^{-2}\right]$$

$$\left\lfloor H_I^{\ D}, X^{D}{}_1\right\rfloor = 0$$

Exact solvability in quantum mechanics

Spectral properties and discretization

Def 8. A quantum mechanical system with Hamiltonian H is called **exactly solvable** if its complete energy spectrum can be calculated algebraically

Its Hilbert space S of bound states consists of a flag of finite dimensional vector spaces

$$S_o \subset S_1 \subset S_2 \ldots \subset S_n \subset \ldots$$

preserved by the Hamiltonian:

$$HS_i \subseteq S_i$$

The bound state eigenfunctions are given by $\psi_n(\vec{x}) = g(\vec{x}) P_n(\vec{s})$

The Hamiltonian can be written as:

$$H = ghg^{-1} \qquad hP_n = E_n P_n$$

$$h = a_i J_i + b_{ij} J^i J^j$$

J_α generate aff(n,R)

Generalized harmonic oscillator

$$V_I = \omega^2\left(x^2 + y^2\right) + \frac{a}{x^2} + \frac{b}{y^2}$$

Gauge factor: $g = x^{p_1} y^{p_2} \exp\left[-\frac{\omega\left(x^2 + y^2\right)}{2}\right]$ $a = p_1\left(p_1 - 1\right)$ $b = p_2\left(p_2 - 1\right)$

After a change of variables, the first superintegrable Hamiltonian becomes

$$h = -2J_3J_1 - 2J_4J_2 + 2J_3 + 2J_4 - \left(2p_1 + 1\right)J_1 - \left(2p_2 + 1\right)J_2$$

where

$$J_1 = \partial_{s_1}, J_2 = \partial_{s_2}, J_3 = s_1\partial_{s_1}, J_4 = s_2\partial_{s_2}, J_5 = s_2\partial_{s_1}, J_6 = s_1\partial_{s_2}$$

It preserves the **flag of polynomials**

$$P_n\left(s_1, s_2\right) = \left\langle \left(s_1\right)^{N_1}\left(s_2\right)^{N_2} \middle\| 0 \le N_1 + N_2 \le n \right\rangle$$

The solutions of the eigenvalue problem are **Laguerre polynomials**

$$HP_{mn} = E_{mn}P_{mn} \qquad P_{mn}\left(x, y\right) = L_n^{\left(-1/2 + p_1\right)}\left(\omega x^2\right)L_m^{\left(-1/2 + p_2\right)}\left(\omega y^2\right)$$

Discretization preserving the H-W algebra

$$h = -2\tilde{J}_3\tilde{J}_1 - 2\tilde{J}_4\tilde{J}_2 + 2\tilde{J}_3 + 2\tilde{J}_4 - (2p_1+1)\tilde{J}_1 - (2p_2+1)\tilde{J}_2$$

$$\tilde{J}_1 = \Delta_{s_1}, \tilde{J}_2 = \Delta_{s_2}, \tilde{J}_3 = (s_1\beta_1)\Delta_{s_1}, \tilde{J}_4 = (s_2\beta_2)\Delta_{s_2}$$

The **commutation relations** between integrals of motion as well as the **spectrum** and the **polynomial solutions** are preseved. No convergence problems arise.

Let us consider a linear spectral problem

$$\begin{tikzcd} L(\partial_x, x)\psi(x) = \lambda\psi(x) \arrow[d] \\ L(\Delta, x\beta)\psi(x\beta) = \lambda\psi(x\beta) \end{tikzcd}$$

$$\psi(x) = \sum_{k=0}^{\infty} a_k x^k \longleftrightarrow \psi(x\beta)\cdot 1 = \sum_{k=0}^{\infty} a_k x_k^{[q]}$$

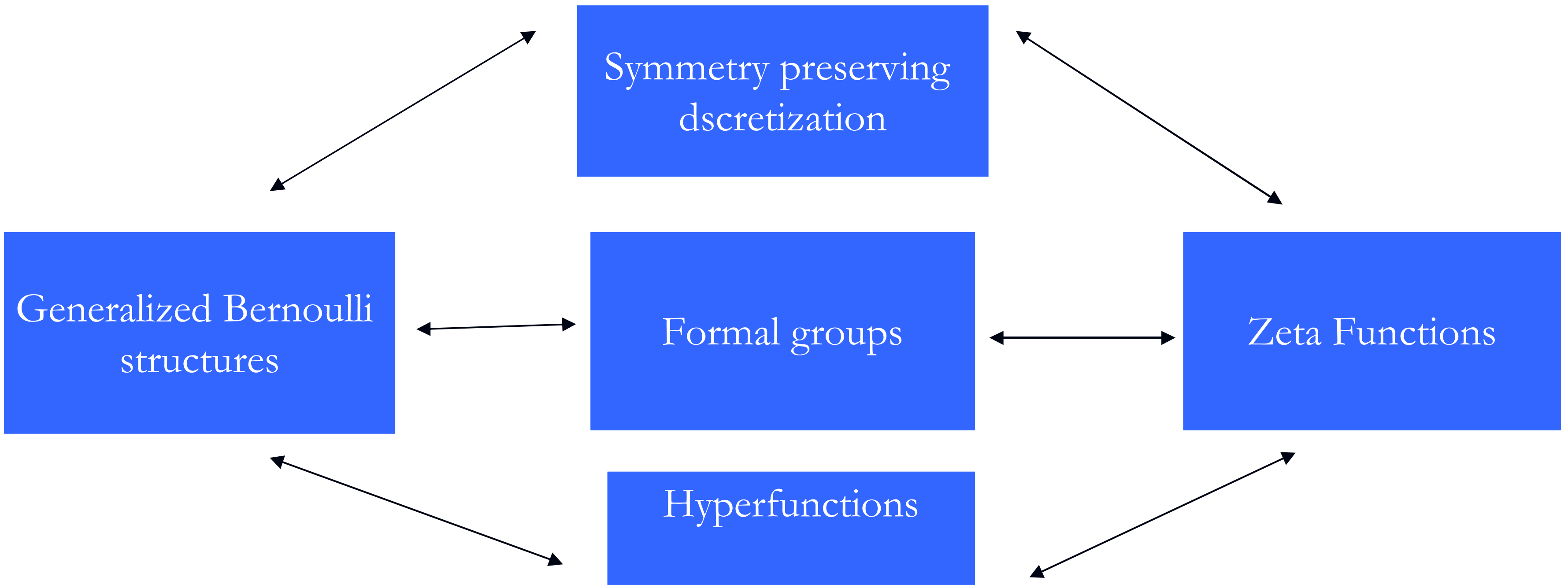
All the discrete versions of the e.s.hamiltonians obtained preserving the Heisenberg-Weyl algebra possess at least formally the same energy spectrum. All the polynomial eigenfunctions can be algebraically computed.

Applications in Algebraic Number Theory:

Generalized Riemann zeta functions
and
New Bernoulli – type Polynomials

Formal groups and finite operator theory

- To each delta operator it corresponds a realization of the **universal formal group law**
- Given a symmetry preserving discretization, we can associate it with a formal group law, a Riemann-type zeta function and a class of Appell polynomials



Symmetry preserving dscretization

Generalized Bernoulli structures

Formal groups

Zeta Functions

Hyperfunctions

Formal groups and number theory

- We will construct L - series attached to formal group exponential laws.
- These series are convergent and generalized the Riemann zeta function
- The Hurwitz zeta function will also be generalized

Theorem 3. Let G(t) be a formal group exponential of the form (2), such that 1/G(t) is a C^∞ function over $\mathbb{R}_+$, rapidly decreasing at infinity.

i) The function

$$L(G,s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{G(t)} t^{s-1} dt,$$

defined for $Re\ s > 1$ admits an holomorphic continuation to the whole $\mathbb{C}$ and, for every $n \in \mathbb{N}$ we have

$$L(G,-n) = (-1)^n \frac{B^G_{n+1}}{n+1} \in \mathbb{Q}[c_1, c_2, ...].$$

ii) Assume that G(t) is of the form (5). For $Re\ s > 1$ the function L(G,s) has a representation in terms of a Dirichlet series

$$L_G = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where the coefficients $a_n \in \mathbb{Q}[c_1, c_2, ...]$ are obtained from the formal expansion

$$\frac{1}{G(t)} = \sum_{n=1}^{\infty} a_n e^{-nt}.$$

iii) Assuming that $G(t) \geq e^t - 1$, the series for L(G,s) is absolutely and uniformly convergent for $Re\ s > 1$ and

$$\left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{Re s}}.$$

Generalized Hurwitz functions

Def. 9 Let G (t) be a formal group exponential of the type (4). The generalized Hurwitz zeta function associated with G is the function *L(G, s, a)* defined for Re s > 1 by

$$L_G(s,a) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{x(1-a)}}{G(x)} x^{s-1} dx = \sum_n^\infty \frac{a_n}{(n+a)^s}$$

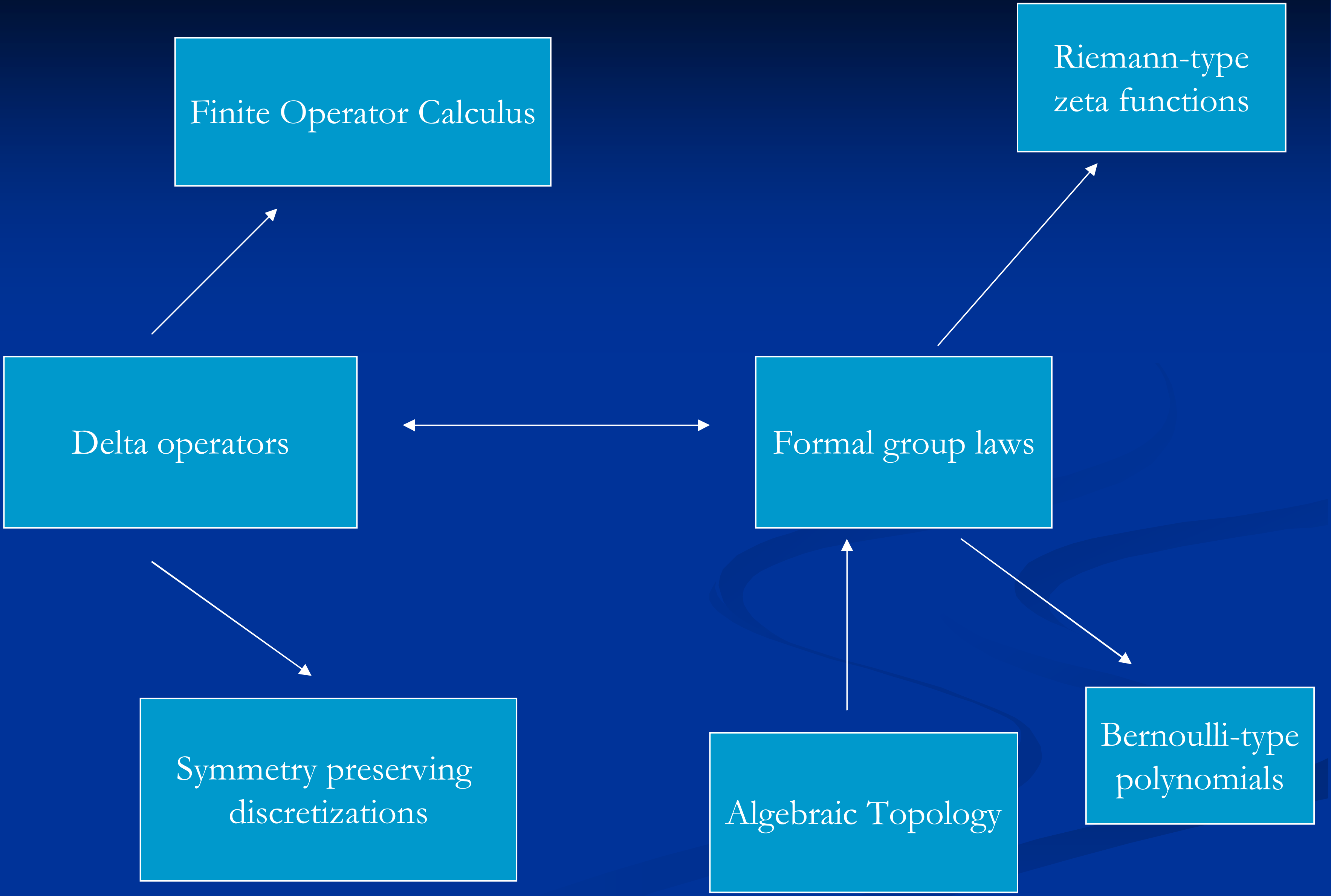
Theorem 4.

$$L_G(-n,a) = -\frac{B^G{}_{n+1}(a)}{n+1}$$

$$\frac{\partial}{\partial a} L_G(s,a) = -sL_G(s+1,a)$$

Lemma 1 (Hasse-type formula):

$$L_G(s,a) = \frac{1}{s-1} \frac{\log(1+\Delta)}{\Delta} a^{1-s} = \frac{1}{s-1} \sum_{n=0}^\infty \frac{(-1)^n}{n1} \Delta^n a^{1-s}$$



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Bernoulli polynomials and numbers

$$\frac{t}{e^t-1}e^{xt}=\sum_{k=0}^{\infty}\frac{B_k(x)}{k!}t^k$$

x = 0 : **Bernoulli numbers** $B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42}, B_8=-\frac{1}{30}$

Fermat’s Last Theorem and class field theory (Kummer)

Theory of Riemann and Riemann-Hurwitz zeta functions

Measure theory in p-adic analysis (Mazur)

Interpolation theory (Boas and Buck)

Combinatorics of groups (V. I. Arnol’d)

Congruences and theory of algebraic equations

Ramanujan identities: QFT and Feynman diagrams

GW invariants, soliton theory (Pandharipande, Veselov)

More than 1500 papers!

Congruences

I. Clausen-von Staudt

If p is a prime number for which p-1 divides k, then

$$B_{2k} + \sum_{p-1|2k} \frac{1}{p} \in Z$$

II. Kummer

Let *m*, *n* be positive even integers such that $m \equiv n \neq 0$ (mod p-1), where p is an odd prime. Then

$$\frac{B_m}{m} \equiv \frac{B_n}{n}, \qquad \mathrm{mod}\ p\,\mathrm{Z}_p$$

Relation with the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1}$$

$$\zeta(1-n) = -\frac{B_n}{n}$$

Hurwitz zeta function:

$$\zeta(s,a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}$$

Integral representation:

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-ax}}{1-e^{-x}} x^{s-1} dx$$

Special values:

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}$$

Universal Bernoulli polynomials

Def. 10. Let $c_1, c_2, \ldots$ be indeterminates over $\mathbf{Q}$. Consider the formal group logarithm

$$F(s) = s + c_1 \frac{s^2}{2} + c_2 \frac{s^3}{3} + \ldots \qquad (1)$$

and the associated formal group exponential

$$G(t) = t - c_1 \frac{t^2}{2} + \left(3c_1^2 - 2c_2\right)\frac{t^3}{6} + \ldots \qquad (2)$$

so that $F(G(t)) = t$ The **universal Bernoulli polynomials** $B_{k,a}^G(x, c_1, \ldots, c_n \ldots) \equiv B_{k,a}^G(x)$

are defined by

$$\left(\frac{t}{G(t)}\right)^a e^{xt} = \sum_{k \geq 0} B_{k,a}^G(x) \frac{t^k}{k!} \qquad (3)$$

Remark. When a = 1 and $c_i = (-1)^i$ then we obtain the classical Bernoulli polynomials

Def. 11. The **universal Bernoulli numbers** are defined by (Clarke)

$$\left(\frac{t}{G(t)}\right)^a = \sum_{k \geq 0} B_{k,a}^G \frac{t^k}{k!} \qquad (4)$$

Properties of UBP

$$B^G_{n.a}(x)=\sum_{k=0}^{n}\binom{n}{k}B^G_{k,a}(0)x^{n-k} \qquad\qquad (x+y)^n=\sum_{k=0}^{n}\binom{n}{k}B^G_{k,a}(x)B^G_{n-k,-a}(y)$$

Generalized Raabe's multiplication theorem

$$B^G_{n+1,a}(x)=\left(x-\frac{G'(t)}{G(t)}\right)B^G_{n,a}(x)$$

Universal Clausen – von Staudt congruence (1990)

If n is even, $$\widehat{B_n}\equiv-\sum_{\substack{p-1|n\\ p\text{prime}}}\frac{c_{p-1}^{n/(p-1)}}{p} \qquad mod \quad \mathbb{Z}[c_1,c_2,\ldots];$$

If n is odd and greater than 1, $$\widehat{B_n}\equiv\frac{c_1^n+c_1^{n-3}c_3}{2} \qquad mod \quad \mathbb{Z}[c_1,c_2,\ldots].$$

Theorem 4. *Let $h\geq 0$, $k>0$, n be integers. Consider the polynomials defined by*

$$\frac{t}{G(t)}e^{xt}=\sum_{k\geq 0}B_k^G(x)\frac{t^k}{k!},$$

Assume that $c_{p-1}\equiv 1$ *mod p for all primes $p\geq 2$. Then*

$$k^n\widetilde{B_n^G}\left(\frac{h}{k}\right)\in\mathbb{Z}[c_1,c_2,\ldots],$$

where $\widetilde{B_n^G}(x)=B_n^G(x)-\widehat{B}_n.$

Conclusions and future perspectives

Main result: correspondence between delta operators, formal groups, symmetry preserving discretizations and algebraic number theory

H-W algebra

Symmetry-preserving discretization of linear PDEs

class of Riemann zeta functions, Hurwitz zeta functions, Appell polynomials of Bernoulli-type

- Finite operator approach for describing symmetries of nonlinear difference equations
- Semigroup theory of linear difference equations and finite operator theory
- **q-estensions of the previous theory**